LOCAL ZETA FUNCTIONS SUPPORTED ON ANALYTIC SUBMANIFOLDS AND NEWTON POLYHEDRA

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Abstract. The local zeta functions (also called Igusa's zeta functions) over p-adic fields are connected with the number of solutions of congruences and exponential sums mod p^m. These zeta functions are defined as integrals over open and compact subsets with respect to the Haar measure. In this paper, we introduce new integrals defined over submanifolds, or more generally, over non-degenerate complete intersection varieties, and study their connections with some arithmetical problems such as estimation of exponential sums mod p^m. In particular we extend Igusa's method for estimating exponential sums mod p^m to the case of exponential sums mod p^m along non-degenerate smooth varieties.

1. Introduction

Let K be a $p-$adic field, i.e. $[K:\mathbb{Q}_p]<\infty$. Let R_K be the valuation ring of K, P_K the maximal ideal of R_K, and $\overline{K}=R_K/P_K$ the residue field of K. The cardinality of the residue field of K is denoted by q, thus $\overline{K}=\mathbb{F}_q$. For $z\in K$, $ord\,(z)\in\mathbb{Z}\cup\{+\infty\}$ denotes the valuation of z, and $|z|_K=q^{-ord(z)}$, ac $z=z\pi^{-ord(z)}$, where π is a fixed uniformizing parameter of R_K.

Let $f_1,\ldots,f_l$ be polynomials in $K\left[x_1,\ldots,x_n\right]$, or, more generally, $K-$analytic functions on an open and compact set $U\subset K^n$. For $1\leq j\leq l$ we define the $K-$analytic set $V^{(j)}(K):=V^{(j)}=\{x\in U\mid f_i\left(x\right)=0,\ 1\leq i\leq j\}$. We assume that $V^{(l-1)}$ is a closed submanifold of U, i.e. $rank_K\left(\frac{\partial f_i}{\partial x_j}(z)\right)=l-1$ for any $z\in V^{(l-1)}$, and that $f_l:V^{(l-1)}\rightarrow K$ is an analytic function on $V^{(l-1)}$. Let $\Phi:K^n\rightarrow\mathbb{C}$ be a Bruhat-Schwartz function (with support in U in the second case). Let ω be a quasicharacter of $K^{\times}$. To these data we associate the following local zeta function:

$$Z_{\Phi}(\omega,f_1,\ldots,f_l,K):=Z_{\Phi}(\omega,V^{(l-1)},f_l)$$
$$=\int\limits_{V^{(l-1)}(K)}\Phi\left(x\right)\omega\left(\ f_l(x)\right)\mid\gamma_{GL}\left(x\right)\mid,$$

for $\omega\in\Omega_0\left(K^{\times}\right)$, where $\mid\gamma_{GL}\left(x\right)\mid$ is the measure induced by a Gel'fand-Leray form on $V^{(l-1)}(K)$ (see Section 2). In this paper we provide a geometric description of the poles of the meromorphic continuation of $Z_{\Phi}(\omega,f_1,\ldots,f_l,K)$ when $f_1,\ldots,f_l$ are non-degenerate with respect to their Newton polyhedra (see Theorem 2 and Remark 8). The relevance of this problem is well-understood in the case in which

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$V^{(l-1)}$ is an open subset of K^n (see e.g. [2], [4], [5], [15], [16], [21], [26], [27], [28], [29], [30], [32]).

The main motivation for this paper is the estimation of exponential sums of the type

$$E(z) := q^{-m(n-l+1)}\sum_{x\in V^{(l-1)}\left(R_K/P_K^m\right)}\Psi\left(zf_{j+1}(x)\right),$$

where $|z|_K = q^m$ with $m \in \mathbb{N}$, $\Psi(\cdot)$ is the standard additive character of K, and $V^{(l-1)}$ is a smooth algebraic variety defined over R_K. In [11], (see also [1], [31]), Katz gave a stationary phase formula for $E(z)$ when f_l has Morse singularities mod P_K. In [18] Moreno proposed to extend Igusa's method (see e.g. [2], [9]) to exponential sums of type $E(z)$.

A more general problem is to estimate oscillatory integrals of type

$$E_\Phi(z, V^{(l-1)}, f_l, K) := E_\Phi(z, V^{(l-1)}, f_l) = \int\limits_{V^{(l-1)}(K)} \Phi(x)\,\Psi(zf_l(x)) \mid \gamma_{GL}(x) \mid,$$

for $|z|_K \gg 0$. Indeed, if $V^{(l-1)}(K)$ has good reduction mod P_K, then $E(z)$ can be expressed as an integral of the previous type (cf. Lemma 4). The relevance of studying integrals of type $E_\Phi(z, V^{(l-1)}, f_l)$ was pointed out in [10] by Kazhdan. In this paper we extend Igusa's method to oscillatory integrals of type $E_\Phi(z, V^{(l-1)}, f_l)$ when f_i, $i = 1, \ldots, l$ are non-degenerate with respect to their Newton polyhedra; more precisely, we show the existence of an asymptotic expansion for $E_\Phi(z, V^{(l-1)}, f_l)$, $|z|_K \gg 0$, which is controlled by the poles of $Z_\Phi(\omega, V^{(l-1)}, f_l)$ (see Theorems 3, 4, Remark 12, and Theorem 6). We are also able to estimate the number of solutions of a polynomial congruence over a smooth algebraic variety (see Theorem 5).

At this point, it is worth to mention that there are other zeta functions supported on analytic sets. The naïve local zeta function supported on a submanifold is defined as

$$Z_\Phi^{\text{naïve}}\left(\omega, V^{(l-1)}, f_l\right) = \int\limits_{K^n} \Phi(x)\,\delta\left(f_1(x), \ldots, f_{l-1}(x)\right)\omega\left(\, f_l(x)\right) \mid dx \mid,$$

for $\omega \in \Omega_0(K^\times)$, where $\delta(\cdot)$ is the Dirac delta function and $\mid dx \mid$ is the Haar measure of K^n normalized so that volume of R_K^n is one. The definition of 'integrals' of type $Z_\Phi^{\text{naïve}}\left(\omega, V^{(l-1)}, f_l\right)$ is based on the fact that an expression of form $\int_{K^n}\Phi(x)\,\delta\left(f_1(x), \ldots, f_{l-1}(x)\right) \mid dx \mid$ gives a well-defined linear functional on the Bruhat-Schwartz space if $V^{(l-1)}$ is a submanifold (see [6] and Section 2.1.1). Furthermore, $Z_\Phi^{\text{naïve}}\left(\omega, V^{(l-1)}, f_l\right) = Z_\Phi(\omega, V^{(l-1)}, f_l)$ (see Lemma 1).

In [7] Hashimoto studied local zeta functions on $\mathbb{R}^n$ and $l = 2$ supported on analytic sets, in the case in which $V^{(1)}(\mathbb{R})$ is a submanifold. Hashimoto showed the existence of an asymptotic expansion for an oscillanting integral supported on $V^{(1)}(\mathbb{R})$ which is controlled by the poles of local zeta functions ([7, Theorem 14]). This result is an extension of Varchenko's result on oscillanting integrals and Newton polyhedra [24]. Hashimoto asserts that a similar result holds when $V^{(1)}(\mathbb{R})$ is singular, more precisely, if $V^{(1)}(\mathbb{R})$ is a non-degenerate complete intersection singularity at the origin ([7, Theorem 27]). However, Hashimoto does not prove that the naïve local zeta functions are 'true integrals' on some half-plane of the complex plane. To the best knowledge of the author, this is a crucial point to establish an asymptotic expansion for oscillanting integrals supported on an analytic

subset. In the real and p-adic cases, when $V^{(l-1)}$ is singular, 'integrals' of type $Z_{\Phi}^{\text{naïve}}\left(\omega, V^{(l-1)}, f_l\right)$ are just 'symbols'. However, under suitable hypotheses on $V^{(l-1)}$ and by using a toroidal resolution of singularities, it is possible to associate to a 'symbol' of type $Z_{\Phi}^{\text{naïve}}\left(\omega, V^{(l-1)}, f_l\right)$ a meromorphic function, which depends on the choice of the resolution.

In the p-adic case, to circumvent the above mentioned problem, we introduce the following local zeta function:

$$\mathcal{Z}_{\Phi}\left(s, V^{(l-1)}, f_l\right) := \lim_{r\to+\infty} \int_{K^n} \Phi(x)\, \delta_r\left(f_1(x), \ldots, f_{l-1}(x)\right) |f_l(x)|_K^s \mid dx \mid$$
$$= \lim_{r\to+\infty} \int_{S_r} \Phi(x)\, q^{r(l-1)} |f_l(x)|_K^s \mid dx \mid,$$

where δ_r is a sequence of functions satisfying $\lim_{r\to+\infty} \delta_r = \delta$, $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 0$, and $S_r := \{x \in K^n \mid ord(f_i(x)) \geqslant r,\ i = 1, \ldots, l-1\}$. The integrals of type $\mathcal{Z}_{\Phi}\left(s, V^{(l-1)}, f_l\right)$ are limits of the integrals considered by Denef in [3].

In the case in which f_i, $i = 1, \ldots, l$ are polynomials defined over R_K which are non-degenerate with respect to their Newton polyhedra mod P_K, we show that $\mathcal{Z}_{\Phi}\left(s, V^{(l-1)}, f_l\right)$ defines a holomorphic function for $\operatorname{Re}(s)$ sufficiently big, in addition, it has a meromorphic continuation to the complex plane as a rational function of q^{-s} which can be computed in terms of the Newton polyhedra of the f_i, see Theorem 7. The zeta functions $\mathcal{Z}_{\Phi}\left(s, V^{(l-1)}, f_l\right)$ may have poles with positive real parts (see Example 8.2), and several examples suggest that the real parts of the poles are eigenvalues of the l-th principal monodromy introduced by Oka (see Example 8.1, Conjecture 1, and references [19]-[20]).

Finally, we want to mention that there are several open questions, problems and conjectures connected with the local zeta functions introduced in this paper. We have formulated some of them along this paper.

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Remark The version to be pusblished in IMRN contains two figures.

2. Local Zeta Functions Supported on Analytic Submanifolds

Let f_i be a polynomial in $K[x_1, \ldots, x_n]$, $f_i(0) = 0$, or, more generally, a $K-$analytic function on an open and compact set $U \subset K^n$, for $i = 1, \ldots, l$. We assume that $2 \leq l \leq n$ and set

$$V^{(j)}(K) = V^{(j)} = \{x \in U \mid f_i(x) = 0,\ 1 \leq i \leq j\},$$

as in the introduction. From now on, we will assume that $V^{(j)}(K)$ is a closed submanifold of U of dimension $n - j$. We refer the reader to [8], [22] for general results on K-analytic manifolds.

Remark 1. *All the K-analytic functions on an open set U which are considered in this paper are given by one power series which is convergent on the whole set U. Since U is totally disconnected, we can take a finite number of open subsets $U_i \subset U$, which form a partition of U, and if we define on each U_i an analytic function by using a power series f_i, we obtain an analytic function on U.*

Let $\Phi : K^n \to \mathbb{C}$ be a Bruhat-Schwartz function (with support in U in the second case). Let $S(K^n)$ be the $\mathbb{C}$-vector space of Bruhat-Schwartz functions.

Let ω be a quasicharacter of $K^\times$, i.e., a continuous homomorphism from $K^\times$ into $\mathbb{C}^\times$. The set of quasicharacters form an Abelian group denoted as $\Omega(K^\times)$. We define an element ω_s of $\Omega(K^\times)$ for every $s \in \mathbb{C}$ as $\omega_s(x) = |x|_K^s = q^{-sord(x)}$. If, for every ω in $\Omega(K^\times)$, we choose $s \in \mathbb{C}$ satisfying $\omega(\pi) = q^{-s}$, then $\omega(x) = \omega_s(x)\chi(ac\, x)$ in which $\chi := \omega\mid_{R_K^\times}$. Hence $\Omega(K^\times)$ is isomorphic to $\mathbb{C}\times\left(R_K^\times\right)^*$, where $\left(R_K^\times\right)^*$ is the group of characters of $R_K^\times$, and $\Omega(K^\times)$ is a one dimensional complex manifold. We note that $\sigma(\omega) := \mathrm{Re}(s)$ depends only on ω, and $|\omega(x)| = \omega_{\sigma(\omega)}(x)$. We define an open subset of $\Omega(K^\times)$ by

$$\Omega_\sigma\left(K^\times\right) = \left\{\omega \in \Omega\left(K^\times\right) \mid \sigma(\omega) > \sigma\right\}.$$

For further details we refer the reader to [9]. To above data we associate the following local zeta function:

$$\begin{aligned} Z_\Phi(\omega, f_1, \ldots, f_l, K) &:= Z_\Phi(\omega, V^{(l-1)}, f_l) \\ &= \int\limits_{V^{(l-1)}(K)} \Phi(x)\,\omega\left(f_l(x)\right) \mid \gamma_{GL}(x) \mid, \end{aligned}$$

for $\omega \in \Omega_0(K^\times)$, where $\mid \gamma_{GL}(x) \mid$ is the measure induced on $V^{(l-1)}(K)$ by a Gel'fand-Leray differential form, i.e., by a form satisfying $\gamma_{GL} \wedge \wedge_{i=1}^{l-1} df_i = \wedge_{i=1}^{n} dx_i$. The Gel'fand-Leray form is not unique, but $Z_\Phi(\omega, V^{(l-1)}, f_l)$ is well-defined because the restriction of γ_{GL} to $V^{(l-1)}$ is independent of the choice of γ_{GL}, see [6, Chap. III, Sect. 1-9]. We warn the reader that $Z_\Phi(\omega, V^{(l-1)}, f_l)$ depends on $f_1, \ldots, f_l$ and not only on $V^{(l-1)}$ and f_l.

Proposition 1. *The zeta function $Z_\Phi(\omega, V^{(l-1)}, f_l)$ is holomorphic on $\Omega_0(K^\times)$.In addition, it has a meromorphic continuation to the whole $\Omega(K^\times)$ as a rational function of $t = \omega(\pi)$. The real parts of the poles of the meromorphic continuation are negative rational numbers.*

Proof. Given a point $b \in V^{(l-1)}(K)$, there exists an open compact subset $W \subseteq K^n$ containing b, and a coordinate system $\phi(x) = (y_1, \ldots, y_n)$, possible after renaming the x_i's, on W such that $y_i = f_i(x)$, $i = 1, \ldots, l-1$, then

$$V^{(l-1)}(K) = \{y_i = 0,\ i = 1, \ldots, l-1\}$$

locally, and $\wedge_{i=1}^n dy_i = J(x) \wedge_{i=1}^n dx_i$ on W, here $J(x)$ is the Jacobian of $\phi(x)$, by schrinking W, if necessary, we may assume that $|J(x)|_K = |J(b)|_K$ for every $x \in W$. By passing to a sufficiently fine disjoint covering of the support of Φ, the zeta function $Z_\Phi(\omega, V^{(l-1)}, f_l)$ becomes a finite sum of Igusa's local zeta functions, more precisely, a finite sum of integrals of type

$$I(\omega) := |J(b)|_K^{-1} \int\limits_{K^{n-l+1}} \Theta(y)\,\omega(h(y)) \left|\bigwedge_{i=l}^{n} dy_i\right|, \tag{2.1}$$

where Θ is a Bruhat-Schwartz function, and $h(y) := \left(f_l \circ \phi^{-1}\right)(0, \ldots, 0, y_l, \ldots, y_n)$ is a K-analytic function defined on an open subset containing the support of Θ. Now the results announced follow from the corresponding results about Igusa's zeta function (see [9, Theorem 8.2.1]). $\square$

In this paper we will study the following problem in a toric setting:

Problem 1. *Provide a geometric description of the poles of the meromorphic continuation of $Z_\Phi(\omega, V^{(l-1)}, f_l)$ in terms of an embedded resolution of singularities of the divisor $D_K := \cup_{i=1}^{l} f_i^{-1}(0)$.*

2.1. The Naïve Definition of $Z_\Phi(\omega, V^{(l-1)}, f_l)$.

2.1.1. *The Dirac Delta function.* Let δ denote the Dirac delta function:

$$\delta(x) = \begin{cases} 0 & \text{if} \quad x \neq 0 \\ \\ +\infty & \text{if} \quad x = 0, \end{cases}$$

for $x \in K^n$. We set for $r \in \mathbb{N}$,

$$\delta_r(x) = \begin{cases} 0 & \text{if} \quad x \notin (\pi^r R_K)^n \\ \\ q^{rn} & \text{if} \quad x \in (\pi^r R_K)^n. \end{cases}$$

Then $\lim_{r\to+\infty} \delta_r(x) = \delta(x)$, and $\int_{K^n} \delta_r(x)\,|dx| = 1$, for any $r \in \mathbb{N}$. We recall that if $\theta \in S(K^n)$, then

$$\lim_{r\to+\infty} \int_{K^n} \theta(x)\,\delta_r(x)\,|dx| = \int_{K^n} \theta(x)\,\delta(x)\,|dx| = \theta(0),$$

i.e., $\lim_{r\to+\infty} \delta_r = \delta$ on $S(K^n)$.

We now review the definition of 'integrals' of type

$$I_\Phi := \int_{K^n} \Phi(x)\,\delta(f_1(x), \ldots, f_{l-1}(x)) \mid dx \mid, \tag{2.2}$$

following Gel'fand and Shilov's book [6], in the case in which $\Phi \in S(K^n)$ and $V^{(l-1)}(K)$ is a submanifold. We may assume without loss of generality that Φ is the product of a constant c by the characteristic function of a ball $W := b + (\pi^{e_0} R_K)^n$, with $b \in V^{(l-1)}(K)$. Since $V^{(l-1)}(K)$ is a submanifold, there exists a coordinate change of the form $y = (y_1, \ldots, y_n) = \phi(x)$, with

$$y_i := \begin{cases} f_i(x) & \text{if} \quad i = 1, \ldots, l-1 \\ \\ x_i - b_i & \text{if} \quad i = l, \ldots, n, \end{cases}$$

such that $\phi : W \to \pi^{d_1} R_K \times \ldots \times \pi^{d_n} R_K$ is a K-analytic isomorphism, for some $(d_1, \ldots, d_n) \in \mathbb{N}^n$, whose Jacobian $J(x)$ satisfies $|J(x)|_K = |J(b)|_K$, for any $x \in W$. By using $y = \phi(x)$ as a change of variables in (2.2) we define

$$\begin{aligned} I_\Phi &= c\,|J(b)|_K^{-1}\, q^{-\sum_{j=l}^{n} d_j} \left(\int_{\pi^{d_1} R_K \times \ldots \times \pi^{d_{l-1}} R_K} \delta(y_1, \ldots, y_{l-1}) \mid dy_1 \ldots dy_{l-1} \mid \right) \\ &= c\,|J(b)|_K^{-1}\, q^{-\sum_{j=l}^{n} d_j}. \end{aligned}$$

The previous definition is independent of the coordinate system used in the calculation because

$$I_\Phi = \int_{V^{(l-1)}(K)} \Phi(x) \mid \gamma_{GL}(x) \mid,$$

where γ_{GL} is a Gel'fand-Leray form on $V^{(l-1)}(K)$. Here we recall that the restriction of γ_{GL} to $V^{(l-1)}(K)$ is unique, and then the previous integral is well-defined [6, Chap. III, Sect. 1-9].

Remark 2. *Let $z \in K$. Since $\Phi(x)\Psi(zf_l(x))$ is a Bruhat-Schwartz function, we can apply this to obtain the following result for oscillatory integrals:*

$$\int_{K^n} \Phi(x)\,\delta(f_1(x),\ldots,f_{l-1}(x))\,\Psi(zf_l(x)) \mid dx \mid = \int_{V^{(l-1)}(K)} \Phi(x)\,\Psi(zf_l(x)) \mid \gamma_{GL}(x) \mid .$$

2.1.2. *The naïve definition.* The naïve local zeta function supported on a submanifold is defined as

$$Z_{\Phi}^{\text{naïve}}(\omega, V^{(l-1)}, f_l) = \int_{K^n} \Phi(x)\,\delta(f_1(x),\ldots,f_{l-1}(x))\,\omega(f_l(x)) \mid dx \mid,$$

for $\omega \in \Omega_0(K^{\times})$.

From the previous discussion about the integrals of type I_{Φ} and by using the same reasoning as in the proof of Proposition 1 and Remark 1 we obtain the following lemma.

Lemma 1. $Z_{\Phi}^{naïve}(\omega, V^{(l-1)}, f_l) = Z_{\Phi}(\omega, V^{(l-1)}, f_l)$, *for $\omega \in \Omega_0(K^{\times})$.*

Let $g_i : U \to K$, $g_i(0) = 0$, $i = 1,\ldots,l$, $l \geq 2$, be an analytic function on an open subset U. We set $W := \{x \in U \mid g_i(x) = 0, i = 1,\ldots,l-1\}$. Another zeta function is defined as follows:

$$\begin{aligned}\mathcal{Z}_{\Phi}(\omega, W, g_l) &:= \lim_{r\to+\infty} \int_{K^n} \Phi(x)\,\delta_r(g_1(x),\ldots,g_{l-1}(x))\,\omega(g_l(x)) \mid dx \mid \\ &= \lim_{r\to+\infty} \int_{S_r} \Phi(x)\,q^{r(l-1)}\omega(g_l(x)) \mid dx \mid,\end{aligned}$$

where $S_r := \{x \in K^n \mid ord(g_i(x)) \geqslant r,\ i = 1,\ldots,l-1\}$. In Section 6 we will study these integrals in a toric setting, in particular we will show the existence of the limit. The $\mathcal{Z}_{\Phi}(s, W, g_l)$ are limits of the integrals considered by Denef in [3]. We emphasize that $Z_{\Phi}^{\text{naïve}}(\omega, W, g_l)$ is not necessary equal to $\mathcal{Z}_{\Phi}(s, W, g_l)$ because $\Phi(x)\,\omega(g_l(x))$ is not a Bruhat-Schwartz function.

3. Polyhedral Subdivisions of $\mathbb{R}_+^n$ and Resolution of Singularities

3.1. **Newton polyhedra.** We set $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geqslant 0\}$. Let G be a non-empty subset of $\mathbb{N}^n$. The *Newton polyhedron* $\Gamma = \Gamma(G)$ associated to G is the convex hull in $\mathbb{R}_+^n$ of the set $\cup_{m\in G}\left(m + \mathbb{R}_+^n\right)$. For instance classically one associates a *Newton polyhedron* $\Gamma(g)$ *(at the origin) to* $g(x) = \sum_m c_m x^m$ ($x = (x_1,\ldots,x_n)$, $g(0) = 0$), being a non-constant polynomial function over K or $K-$analytic function in a neighborhood of the origin, where $G = \text{supp}(g) := \{m \in \mathbb{N}^n \mid c_m \neq 0\}$. Further we will associate more generally a Newton polyhedron to an analytic mapping.

We fix a Newton polyhedron Γ as above. We first collect some notions and results about Newton polyhedra that will be used in the next sections. Let $\langle\cdot,\cdot\rangle$ denote the usual inner product of $\mathbb{R}^n$, and identify the dual space of $\mathbb{R}^n$ with $\mathbb{R}^n$ itself by means of it.

For $a \in \mathbb{R}_+^n$, we define

$$d(a, \Gamma) = \min_{x \in \Gamma} \langle a, x \rangle \,,$$

and *the first meet locus* $F(a, \Gamma)$ of a as

$$F(a, \Gamma) := \{x \in \Gamma \mid \langle a, x \rangle = d(a, \Gamma)\}.$$

The first meet locus is a face of Γ. Moreover, if $a \neq 0$, $F(a, \Gamma)$ is a proper face of Γ.

If $\Gamma = \Gamma(g)$, we define the *face function* $g_a(x)$ *of* $g(x)$ *with respect to* a as

$$g_a(x) = g_{F(a,\Gamma)}(x) = \sum_{m \in F(a,\Gamma)} c_m x^m.$$

In the case of functions having subindices, say $g_i(x)$, we will use the notation $g_{i,a}(x)$ for the face function of $g_i(x)$ with respect to a.

We will say that $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ is *a positive vector* (respectively *strictly positive vector*), if $a_i \geq 0$, for $i = 1, \ldots, n$, (respectively if $a_i > 0$, for $i = 1, \ldots, n$). We use the notation $a \succ 0$ to mean that a is strictly positive.

3.2. **Polyhedral Subdivisions Subordinate to a Polyhedron.** We define an equivalence relation in $\mathbb{R}_+^n$ by taking $a \sim a' \Leftrightarrow F(a, \Gamma) = F(a', \Gamma)$. The equivalence classes of $\sim$ are sets of the form

$$\Delta_\tau = \{a \in \mathbb{R}_+^n \mid F(a, \Gamma) = \tau\},$$

where τ is a face of Γ.

We recall that the cone strictly spanned by the vectors $a_1, \ldots, a_r \in \mathbb{R}_+^n \setminus \{0\}$ is the set $\Delta = \{\lambda_1 a_1 + \ldots + \lambda_r a_r \mid \lambda_i \in \mathbb{R}_+, \, \lambda_i > 0\}$. If $a_1, \ldots, a_r$ are linearly independent over $\mathbb{R}$, Δ is called a *simplicial cone*. If $a_1, \ldots, a_r \in \mathbb{Z}^n$, we say Δ is a *rational cone*. If $\{a_1, \ldots, a_r\}$ is a subset of a basis of the $\mathbb{Z}$-module $\mathbb{Z}^n$, we call Δ a *simple cone*.

A precise description of the geometry of the equivalence classes modulo $\sim$ is as follows. Each *facet* (i.e. a face of codimension one) γ of Γ has a unique vector $a(\gamma) = (a_{\gamma,1}, \ldots, a_{\gamma,n}) \in \mathbb{N}^n \setminus \{0\}$, whose nonzero coordinates are relatively prime, which is perpendicular to γ. We denote by $\mathfrak{D}(\Gamma)$ the set of such vectors. The equivalence classes are rational cones of the form

$$\Delta_\tau = \{\sum_{i=1}^{r} \lambda_i a(\gamma_i) \mid \lambda_i \in \mathbb{R}_+, \, \lambda_i > 0\},$$

where τ runs through the set of faces of Γ, and γ_i, $i = 1, \ldots, r$ are the facets containing τ. We note that $\Delta_\tau = \{0\}$ if and only if $\tau = \Gamma$. The family $\{\Delta_\tau\}_\tau$, with τ running over the proper faces of Γ, is a partition of $\mathbb{R}_+^n \setminus \{0\}$; we call this partition a *polyhedral subdivision of* $\mathbb{R}_+^n$ *subordinate* to Γ. We call $\left\{\overline{\Delta_\tau}\right\}_\tau$, the family formed by the topological closures of the Δ_τ, a *fan subordinate* to Γ.

Each cone Δ_τ can be partitioned into a finite number of simplicial cones $\Delta_{\tau,i}$. In addition, the subdivision can be chosen such that each $\Delta_{\tau,i}$ is spanned by part of $\mathfrak{D}(\Gamma)$. Thus from the above considerations we have the following partition of $\mathbb{R}_+^n \setminus \{0\}$:

$$\mathbb{R}_+^n \setminus \{0\} = \bigcup_\tau \left(\bigcup_{i=1}^{l_\tau} \Delta_{\tau,i} \right),$$

where τ runs over the proper faces of Γ, and each $\Delta_{\tau,i}$ is a simplicial cone contained in Δ_τ. We will say that $\{\Delta_{\tau,i}\}$ is a *simplicial polyhedral subdivision of* $\mathbb{R}_+^n$ *subordinate* to Γ, and that $\left\{\overline{\Delta}_{\tau,i}\right\}$ is a *simplicial fan subordinate* to Γ.

By adding new rays , each simplicial cone can be partitioned further into a finite number of simple cones. In this way we obtain a *simple polyhedral subdivision* of $\mathbb{R}_+^n$ *subordinate* to Γ, and a *simple fan subordinate* to Γ (or a *complete regular fan*) (see e.g. [12]).

Given a rational polyhedral subdivision Σ^* of $\mathbb{R}_+^n$, we denote by $\mathrm{Vert}(\Sigma^*)$ the set of all the generators of the cones in Σ^*(this set is also called *the skeleton of* Σ^*). Each n-dimensional simple cone in Σ^* generated, say, by $a_i = (a_{i,1}, \ldots, a_{i,n}) \in \mathrm{Vert}(\Sigma^*)$, $i = 1, \ldots, n$, can be identified with a unimodular matrix $[a_1, \ldots, a_n]$.

3.3. **The Newton polyhedron associated to an analytic mapping.** Let $\boldsymbol{f} = (f_1, \ldots, f_l)$, $\boldsymbol{f}(0) = 0$, be a non-constant polynomial mapping, or more generally, an analytic mapping defined on a neighborhood $U \subseteq K^n$ of the origin. In this paper we associate to $\boldsymbol{f}$ a Newton polyhedron $\Gamma(\boldsymbol{f}) := \Gamma\left(\prod_{i=1}^{l} f_i(x)\right)$. From a geometrical point of view, $\Gamma(\boldsymbol{f})$ is the Minkowski sum of the $\Gamma(f_i)$, for $i = 1, \cdots, l$, (see e.g. [20], [23]). By using the results previously presented, we can associate to $\Gamma(\boldsymbol{f})$ a simple (or simplicial) polyhedral subdivision $\Sigma^*(\boldsymbol{f})$ of $\mathbb{R}_+^n$ subordinate to $\Gamma(\boldsymbol{f})$. The polyhedron $\Gamma(\boldsymbol{f})$ is useful to construct $\Sigma^*(\boldsymbol{f})$, and then the corresponding toric manifold. However, if $l > 1$, the description of the real parts of the poles of the local zeta functions defined in the introduction requires $\Gamma(\boldsymbol{f})$, and $\Gamma(f_j)$, $j = 1, \ldots, l$, as we will see later on.

Remark 3. *A basic fact about the Minkowski sum operation is the additivity of the faces. From this fact follows:*
(1) $F(a, \Gamma(\boldsymbol{f})) = \sum_{j=1}^{l} F(a, \Gamma(f_j))$, *for* $a \in \mathbb{R}_+^n$ *;*
(2) $d(a, \Gamma(\boldsymbol{f})) = \sum_{j=1}^{l} d(a, \Gamma(f_j))$, *for* $a \in \mathbb{R}_+^n$ *;*
(3) let τ *be a proper face of* $\Gamma(\boldsymbol{f})$, *and let* τ_j *be proper face of* $\Gamma(f_j)$, *for* $i = 1, \cdots, l$. *If* $\tau = \sum_{j=1}^{l} \tau_j$, *then* $\Delta_\tau \subseteq \overline{\Delta}_{\tau_j}$, *for* $i = 1, \cdots, l$.

Remark 4. *Note that the equivalence relation,*

$$a \sim a' \Leftrightarrow F(a, \Gamma(\boldsymbol{f})) = F(a', \Gamma(\boldsymbol{f})),$$

used in the construction of a polyhedral subdivision of $\mathbb{R}_+^n$ *subordinate to* $\Gamma(\boldsymbol{f})$ *can be equivalently defined in the following form:*

$$a \sim a' \Leftrightarrow F(a, \Gamma(f_j)) = F(a', \Gamma(f_j)), \text{ for each } j = 1, \ldots, l.$$

This last definition is used in Oka's book [20].

3.4. **Non-degeneracy Conditions.**

Definition 1. *(1) Let* $\boldsymbol{f} = (f_1, \ldots, f_l)$, $\boldsymbol{f}(0) = 0$, *be an analytic mapping defined on a neighborhood* $U \subseteq K^n$ *of the origin. Let* $\Gamma_i := \Gamma(f_i)$ *be the Newton polyhedron of* f_i *at the origin, for* $i = 1, \ldots, l$. *The mapping* $\boldsymbol{f}$ *is called non-degenerate with respect to* $(\Gamma_1, \ldots, \Gamma_l)$ *at the origin (or simply non-degenerate), if for every strictly positive vector* $a \in \mathbb{R}^n$ *and any*

$$z \in \left\{z \in \left(K^\times\right)^n \cap U \mid f_{1,a}(z) = \ldots = f_{l,a}(z) = 0\right\},$$

it satisfies that $\mathrm{rank}_K \left[\frac{\partial f_{i,a}}{\partial x_j}(z)\right] = \min\{l, n\}$.

(2) Let $V^{(j)}(K) = V^{(j)} = \{z \in U \mid f_1(z) = \ldots = f_j(z) = 0\}$*, for* $j = 1, \ldots, l$*, as before. If the analytic mapping*

$$\begin{array}{ccc} U & \to & K^j \\ x & \to & (f_1(x), \ldots, f_j(x)) \end{array}$$

is non-degenerate, we will say that $V^{(j)}$ *is a germ of non-degenerate variety.*
(3) Let $\boldsymbol{f} = (f_1, \ldots, f_l)$*,* $\boldsymbol{f}(0) = 0$*, be a non-constant polynomial mapping. Let* $\Gamma_i = \Gamma(f_i)$ *be the Newton polyhedron of* f_i *at the origin, for* $i = 1, \ldots, l$*. The mapping* $\boldsymbol{f}$ *is called strongly non-degenerate with respect to* $(\Gamma_1, \ldots, \Gamma_l)$ *(or simply strongly non-degenerate), if for every positive vector* $a \in \mathbb{R}^n$*, including the origin, and any* $z \in \left\{z \in (K^\times)^n \mid f_{1,a}(z) = \ldots = f_{l,a}(z) = 0\right\}$*, it satisfies that* $\mathrm{rank}_K \left[\frac{\partial f_{i,a}}{\partial x_j}(z)\right] = \min\{l, n\}$.
(4) Let $V^{(j)}(K) = V^{(j)} = \{z \in K^n \mid f_1(z) = \ldots = f_j(z) = 0\}$*, for* $j = 1, \ldots, l$*. If the mapping* $x \to (f_1(x), \ldots, f_j(x))$ *is strongly non-degenerate, we will say that* $V^{(j)}$ *is a non-degenerate complete intersection variety with respect to the coordinate system* $x = (x_1, \ldots, x_n)$*.*

The above notion was introduced by Khovansky [13], see also [20]. For a discussion about the relation between Khovansky's non-degeneracy notion and other similar notions we refer the reader to [28].

Remark 5. *We want to keep the classical terminology used in singularity theory. However, we will use this terminology in a more general setting. We will use the following conventions. By a* K*-algebraic variety* W*, always affine in this paper, we mean the set of* K*-rational points of* W *with the structure of* K*-analytic set. By a smooth* K*-algebraic variety* V*, we mean the set of* K*-rational points of* V *with the structure of* K*-analytic closed submanifold. In particular the dimension of* V *is the dimension of the underlying* K*-analytic submanifold.*

Definition 2. *An analytic function* $f_i(x)$ *is called convenient if for any* $k = 1, \ldots, n$*, there exists a monomial* $x_k^{m_k}$ *with non-zero coefficient in the Taylor expansion of* $f_i(x)$*. An analytic mapping*

$$\begin{array}{cccc} (f_1, \ldots, f_j): & U & \to & K^j \\ & x & \to & (f_1(x), \ldots, f_j(x)) \end{array}$$

is called convenient, if each $f_i(x)$ *is convenient for* $i = 1, \ldots, j$*. In this case we will say that* $V^{(j)}$ *is convenient variety.*

We set $E_i := \left(0, \ldots, \underbrace{\begin{array}{c} 1 \\ i - \text{th place} \end{array}}, \ldots, 0\right)$, for $i = 1, \ldots, n$. From now on, we will assume that $\boldsymbol{f} = (f_1, \ldots, f_l)$ is convenient, thus there exists a simple polyhedral subdivision of $\mathbb{R}^n_+$ subordinate to $\Gamma(\boldsymbol{f})$ such that any not strictly positive vector in $\mathrm{Vert}(\Sigma^*)$ belongs to $\{E_1, \ldots, E_n\}$.

3.5. **Resolution of Singularities of Non-degenerate Complete Intersection Varieties.** The resolution of singularities for non-degenerate complete intersection varieties is a well-known fact (see e.g. [13], [17], [20], [24], see also [28], and references therein). The version of the resolution of singularities needed here is a simple variation of the one given in [20, Theorem 3.4], [17, Proposition 2.5]. For the material needed to adapt the proof given in Oka´s book to the p-adic setting we refer the

reader to [9], [22]. We now review the basic facts about resolution of singularities of non-degenerate complete intersection varieties, without proofs, following Oka's book. We warn the reader that our terminology and notation are slightly different to the corresponding in [20].

Let $V^{(j)}(K)$, $j=l-1,l$, be germs of convenient and non-degenerate complete intersection varieties. Fix a rational simple polyhedral subdivision $\Sigma^{*}=\Sigma^{*}(\boldsymbol{f})$ of $\mathbb{R}_{+}^{n}$ subordinate to $\Gamma(\boldsymbol{f})$. Let $X(K)$ be the toric manifold corresponding to Σ^{*}, and let $h:X(K)\rightarrow U$ be the corresponding toric modification. We set $(V^{*})^{(j)}(K):=V^{(j)}(K)\cap(K^{\times})^{n}$, $j=l-1,l$, and $V_{U}^{(j)}(K)$ for the closure of $(V^{*})^{(j)}(K)$ in U, $j=l-1,l$. Let $\widetilde{V}^{(j)}(K)$ be the strict transform of $(V^{*})^{(j)}(K)$ in $X(K)$, for $j=l-1,l$. We set $H^{(j)}(K):=\{z\in U\mid f_{j}(z)=0\}$, $(H^{*})^{(j)}(K):=H^{(j)}(K)\cap(K^{\times})^{n}$, and $H_{U}^{(j)}(K)$ for the closure of $(H^{*})^{(j)}(K)$ in U, for $j=1,\ldots,l$. Let $\widetilde{H}^{(j)}(K)$ be the strict transform of $(H^{*})^{(j)}(K)$ in $X(K)$, for $j=1,\ldots,l$.

We recall that $h:X(K)\smallsetminus h^{-1}(0)\rightarrow U\smallsetminus\{0\}$ is a K-analytic isomorphism, and that

$$h^{-1}(0)=\bigcup_{\substack{a\in\mathrm{Vert}(\Sigma^{*})\\ a\succ 0}}E_{a}(K),$$

where $E_{a}(K)$ is the compact exceptional divisor corresponding to a, see [20, Corollary 1.4.1].

Theorem 1 ([20, Theorem 3.4])**.** *Assume that $V^{(l)}(K)$, $V^{(l-1)}(K)$ are germs of convenient and non-degenerate complete intersection varieties at the origin. Fix a rational simple polyhedral subdivision $\Sigma^{*}=\Sigma^{*}(\boldsymbol{f})$ of $\mathbb{R}_{+}^{n}$ subordinate to $\Gamma(\boldsymbol{f})$. Let $X(K)$ be the toric manifold corresponding to Σ^{*}, and let $h:X(K)\rightarrow U$ be the corresponding toric modification. There exist a neighborhood $U_{0}\subseteq U$ of the origin such that the following assertions are true over U_{0}. Put $X'(K):=X(K)\smallsetminus h^{-1}(0)$ and $Y'(K):=U\smallsetminus\{0\}$.*
(1) $h:\widetilde{V}^{(j)}(K)\rightarrow V_{U}^{(j)}(K)$ is a proper mapping such that $h:\widetilde{V}^{(j)}(K)\cap X'(K)\rightarrow V_{U}^{(j)}(K)\cap Y'(K)$ is a K-analytic isomorphism, for $j=l-1,l$.
(2) The divisor of the pullback function $h^{*}f_{j}$ is given by

$$div\left(h^{*}f_{j}\right)(K)=\widetilde{H}^{(j)}(K)+\sum_{\substack{a\in Vert(\Sigma^{*})\\ a\succ 0}}d\left(a,\Gamma_{j}\right)E_{a}(K).$$

(3) $\widetilde{V}^{(j)}(K)$ is a submanifold of $X(K)$ of dimension $n-j$ so that $\widetilde{V}^{(j)}(K)=\cap_{i=1}^{j}\widetilde{H}^{(i)}(K)$, $j=l-1,l$. In addition $\widetilde{V}^{(j)}(K)$, $j=l-1,l$, intersect the exceptional divisor of h transversely.
(4) $h:\widetilde{V}^{(l-1)}(K)\smallsetminus h^{-1}(0)\rightarrow V^{(l-1)}(K)\smallsetminus\{0\}$ is a K-analytic isomorphism.

Remark 6. *In the next section we use the previous theorem to give an explicit list for the possible poles of $Z_{\Phi}(\omega,V^{(l-1)},f_{l})$. We fix some notations needed for the next section. Let $\Delta\in\Sigma^{*}$ be an n-dimensional simple cone generated by $a_{i}=(a_{1,i},\ldots,a_{n,i})$, $i=1,\ldots,n$. Then in the chart of $X(K)$ corresponding to Δ, the map h has the form*

$$\begin{array}{cccc} h: & K^{n} & \rightarrow & U\\ & y & \rightarrow & x, \end{array}$$

where $x_{i}=\prod_{j=1}^{n}y_{j}^{a_{i,j}}$, with $[a_{i,j}]=[a_{1},\ldots,a_{n}]$. Denote this chart by W_{Δ}.

We note that since h only maps to U instead to the whole K^n, at some charts it will not be defined everywhere on K^n. We set $I = \{1, \ldots, r\}$ with $r \leq n$, and

$$T_I(K) := \{y \in W_\Delta \mid y_i = 0 \Longleftrightarrow i \in I\}.$$

We consider on W_Δ points on $h^{-1}(0)$ different from the origin of W_Δ. Let $b \in T_I(K)$ be one of these points. We consider the following two cases: (I) $b \in T_I(K) \cap \widetilde{V}^{(l)}(K)$; (II) $b \in T_I(K) \cap \widetilde{V}^{(l-1)}(K)$ and $b \notin \widetilde{V}^{(l)}(K)$.

Case I. In this case $b = (0, \ldots, 0, b_{r+1}, \ldots, b_n)$ with

$$\widetilde{b} = (b_{r+1}, \ldots, b_n) \in (K^\times)^{n-r}.$$

By Theorem 1(2),

$$(f_i \circ h)(y) = \left(\prod_{j=1}^{r} y_j^{d(a_j, \Gamma_i)}\right)\left(\widetilde{f}_i(y_{r+1}, \ldots, y_n) + O_i(y_1, \ldots, y_n)\right), \tag{3.1}$$

where the $O_i(y_1, \ldots, y_n)$ are analytic functions belonging to the ideal generated by $y_1, \ldots, y_r$. By using that

$$\widetilde{b} \in \left(K^\times\right)^{n-r} \cap \left\{\widetilde{f}_1(b_{r+1}, \ldots, b_n) = \cdots = \widetilde{f}_l(b_{r+1}, \ldots, b_n) = 0\right\}, \tag{3.2}$$

and Theorem 1(3), there exists a coordinate system

$$y' = (y_1, \ldots, y_r, y'_{r+1}, \ldots, y'_n)$$

in a neighborhood W_b of b such that

$$(f_i \circ h)(y') = \left(\prod_{j=1}^{r} y_j^{d(a_j, \Gamma_i)}\right) y'_{r+i}, \tag{3.3}$$

for $i = 1, \ldots, l$, and

$$\left\{y'_{r+1} = \ldots = y'_{r+l} = 0\right\}, \text{ respectively } \left\{y'_{r+1} = \ldots = y'_{r+l-1} = 0\right\},$$

is a local description in W_b of $\widetilde{V}^{(l)}(K)$, respectively of $\widetilde{V}^{(l-1)}(K)$.

Case II. This case is similar to the previous one except that

$$\widetilde{b} \in \left(K^\times\right)^{n-r} \cap \left\{\widetilde{f}_1(b_{r+1}, \ldots, b_n) = \cdots = \widetilde{f_{l-1}}(b_{r+1}, \ldots, b_n) = 0\right\},$$

and (3.3) holds for $i = 1, \ldots, l-1$, and

$$(f_l \circ h)(y') = \left(\prod_{j=1}^{r} y_j^{d(a_j, \Gamma_l)}\right) u(y'), \tag{3.4}$$

with $|u(y')|_K = |u(b)|_K$ for any $y' \in W_b$, and $\left\{y'_{r+1} = \ldots = y'_{r+l-1} = 0\right\}$ is a local description in W_b of $\widetilde{V}^{(l-1)}(K)$.

Remark 7. *(1) If we replace in Theorem 1 the condition "$V^{(l)}(K)$, $V^{(l-1)}(K)$ are germs of convenient non-degenerate complete intersection varieties at the origin,"by "$V^{(l)}(K)$, $V^{(l-1)}(K)$ are convenient and non-degenerate complete intersection varieties," and U by K^n, with a similar proof we obtain a global version of Theorem 1, that is, the conclusions (1)-(4) are valid without the condition "there exists a neighborhood $U_0 \subset U$ of the origin." In this case $\widetilde{V}^{(l)}(K)$ and $\widetilde{V}^{(l-1)}(K)$ may have components that are disjoint with the exceptional divisor of h.*

(2) The condition "there exists a neighborhood $U_0 \subset U$ of the origin" may be replaced by "U is a sufficiently small neighborhood of the origin."

4. The Poles of the Meromorphic Continuation of $Z_\Phi(\omega, V^{(l-1)}, f_l)$

Given $a = (a_1, \ldots, a_n) \in \mathbb{R}^n \setminus \{0\}$, we put $\sigma(a) := a_1 + \ldots + a_n$ and $d(a, \Gamma_l) = \min_{m \in \Gamma_l} \langle m, a \rangle$ as before. If $d(a, \Gamma_l) \neq 0$, we define

$$\mathcal{P}(a) = \left\{ -\left(\frac{\sigma(a) - \sum_{j=1}^{l-1} d(a, \Gamma_j)}{d(a, \Gamma_l)} \right) + \frac{2\pi\sqrt{-1}k}{d(a, \Gamma_l) \log q}, \ k \in \mathbb{Z} \right\}.$$

Theorem 2. *Let $\boldsymbol{f} = (f_1, \ldots, f_l) : U \to K^l$, $l \geq 2$, $\boldsymbol{f}(0) = 0$, be an analytic mapping defined on a neighborhood $U \subseteq K^n$ of the origin. Assume that $V^{(l)}(K)$ and $V^{(l-1)}(K)$ are germs of convenient and non-degenerate complete intersection varieties, and that $V^{(l-1)}(K)$ is a closed submanifold of U. Fix a rational simple polyhedral subdivision $\Sigma^* = \Sigma^*(\boldsymbol{f})$ of $\mathbb{R}_+^n$ subordinate to $\Gamma(\boldsymbol{f})$. If U is sufficiently small and Φ is a Bruhat-Schwartz function whose support is contained in U, then $Z_\Phi(\omega, V^{(l-1)}, f_l)$ is a rational function of $t = \omega(\pi)$, and its poles belong to the set*

$$\bigcup_{\substack{a \in Vert(\Sigma^*) \\ a \succ 0}} \mathcal{P}(a) \cup \left\{ -1 + \frac{2\pi\sqrt{-1}k}{\log q}, \ k \in \mathbb{Z} \right\}.$$

Proof. We fix a rational simple polyhedral subdivision $\Sigma^* = \Sigma^*(\boldsymbol{f})$ of $\mathbb{R}_+^n$, and use the notation introduced in Theorem 1 and Remark 6. By Theorem 1 (4),

$$h : \widetilde{V}^{(l-1)}(K) \smallsetminus h^{-1}(0) \to V^{(l-1)}(K) \smallsetminus \{0\} \tag{4.1}$$

is a K-analytic isomorphism, where $h^{-1}(0)$ is the exceptional divisor of h. We use (4.1) as a change of variables in $Z_\Phi(\omega, V^{(l-1)}, f_l)$:

$$\begin{aligned} Z_\Phi(\omega, V^{(l-1)}, f_l) &= \int_{V^{(l-1)}(K) \setminus \{0\}} \Phi(x)\, \omega(f_l(x)) \mid \gamma_{GL}(x) \mid \\ &= \int_{\widetilde{V}^{(l-1)}(K) \smallsetminus h^{-1}(0)} \Phi^*(y)\, \omega(f_l^*(y)) \mid \gamma_{GL}^*(y) \mid \end{aligned}$$

for $\omega \in \Omega_0(K^\times)$, where $\mid \gamma_{GL}(x) \mid$ is the measure induced on $V^{(l-1)}(K)$ by a Gel'fand-Leray differential form, and $\gamma_{GL}^*(y)$ is the pullback of $\gamma_{GL}(x)$ by h.

Since $\Phi^*(y)$ has compact support, it is sufficient to establish the theorem for integrals of type

$$I(\omega) := \int_{\widetilde{V}^{(l-1)}(K) \smallsetminus h^{-1}(0)} \Theta(y)\, \omega(f_l^*(y)) \mid \gamma_{GL}^*(y) \mid,$$

where $\Theta(y)$ is the characteristic function of a neighborhood W_b (an open compact set which may be schrinked when necessary) of a point $b \in \widetilde{V}^{(l-1)}(K)$. Furthermore, we may assume that $W_b = c + \pi^e R_K^n$, for some $c = (c_1, \ldots, c_n) \in K^n$, and that $W_b \subset W_\Delta$, the chart corresponding to a simple cone $\Delta \in \Sigma^*$ generated by $a_i =$

$(a_{1,i},\ldots,a_{n,i})$, $i=1,\ldots,n$, as in Remark 6. We take $I=\{1,\ldots,r\}$ with $r\leq n$, and study the meromorphic continuation of $I(\omega)$ for the two cases considered in Remark 6.

Case I ($b\in T_I(K)\cap\widetilde{V}_l(K)$). In this case $r+l\leq n$, and there exists a coordinate system $y'=\left(y_1,\ldots,y_r,y'_{r+1},\ldots,y'_n\right)$ in W_b such that

$$(f_i\circ h)(y')=\left(\prod_{j=1}^{r}y_j^{d(a_j,\Gamma_i)}\right)y'_{r+i},$$

for $i=1,\ldots,l$, $\widetilde{V}^{(l-1)}(K)=\left\{y'_{r+1}=\ldots=y'_{r+l-1}=0\right\}$, and

$$h^*\left(\bigwedge_{1\leq j\leq n}dx_j\right)=\left(\eta(y)\prod_{j=1}^{r}y_j^{\sigma(a_j)-1}\right)\bigwedge_{1\leq j\leq r}dy_j\wedge\bigwedge_{r+1\leq j\leq n}y'_j,$$

where $\eta(y)$ is a unit of the local ring $\mathcal{O}_{X(K),b}$. By schrinking W_b if necessary, we assume that $|\eta(y)|_K=|\eta(b)|_K$, for any $y\in W_b$. The form γ^*_{GL} on $\widetilde{V}^{(l-1)}(K)$ is determined by the condition

$$\gamma^*_{GL}(y')\wedge\bigwedge_{i=1}^{l-1}d\left(\left(\prod_{j=1}^{r}y_j^{d(a_j,\Gamma_i)}\right)y'_{r+i}\right)$$
$$=\left(\prod_{j=1}^{r}y_j^{\sigma(a_j)-1}\right)\bigwedge_{1\leq j\leq r}dy_j\wedge\bigwedge_{r+1\leq j\leq n}dy'_j.$$

Since the left side of the previous formula equals

$$\left(\prod_{j=1}^{r}y_j^{\sum_{i=1}^{l-1}d(a_j,\Gamma_i)}\right)\gamma^*_{GL}(y')\wedge\bigwedge_{i=1}^{l-1}dy'_{r+i}$$

on $\widetilde{V}^{(l-1)}(K)$, we can take $\gamma^*_{GL}(y')$ equal to

$$\left(\prod_{j=1}^{r}y_j^{\sigma(a_j)-1-\sum_{i=1}^{l-1}d(a_j,\Gamma_i)}\right)\bigwedge_{1\leq j\leq r}dy_j\wedge dy'_{r+l}\wedge\bigwedge_{r+l+1\leq j\leq n}dy'_j \tag{4.2}$$

which is an analytic form, and then

$$\sigma(a_j)-1-\sum_{i=1}^{l-1}d(a_j,\Gamma_i)\geq 0,\ \text{for } j=1,\ldots,r. \tag{4.3}$$

We set $Y(K):=W_b\cap\left(\widetilde{V}^{(l-1)}(K)\smallsetminus h^{-1}(0)\right)$,

$$\widetilde{\chi}(y'):=\chi\left(ac\ \left(\prod_{j=1}^{r}y_j^{d(a_j,\Gamma_l)}\right)\right)\chi\left(ac\ y'_{r+l}\right),$$

and

$$|dy''|:=\left|\bigwedge_{j=1}^{r}dy_j\wedge dy'_{r+l}\wedge\bigwedge_{j=r+l+1}^{n}dy'_j\right|.$$

We can express $I(\omega)$ as follows:

$$\int\limits_{Y(K)} \widetilde{\chi}(y') \prod_{j=1}^{r} \left(|y_j|_K^{d(a_j,\Gamma_l)s+\sigma(a_j)-1-\sum_{i=1}^{l-1} d(a_j,\Gamma_i)} \right) |y'_{r+l}|_K^s \, |dy''| \,,$$

i.e., $I(\omega)$ is the product (up to a constant) of the following two integrals:

$$\prod_{j=1}^{r} \int\limits_{c_j+\pi^e R_K} \chi\left(ac\ y_j\right)^{d(a_j,\Gamma_l)} |y_j|_K^{d(a_j,\Gamma_l)s+\sigma(a_j)-1-\sum_{i=1}^{l-1} d(a_j,\Gamma_i)} |dy_j| \,, \tag{4.4}$$

and

$$\int\limits_{c_{r+l}+\pi^e R_K} \chi\left(ac\ y'_{r+l}\right) \left|y'_{r+l}\right|_K^s \left|dy'_{r+l}\right| , \tag{4.5}$$

for $\mathrm{Re}(s) > 0$. By applying Lemma 8.2.1 in [9] we obtain the meromorphic continuation for integrals in (4.4) and (4.5). Therefore the real parts of the poles of the meromorphic continuation of $Z_\Phi(\omega, V^{(l-1)}, f_l)$ have the form

$$-\left(\frac{\sigma\left(a\right) - \sum_{j=1}^{l-1} d\left(a, \Gamma_j\right)}{d\left(a, \Gamma_l\right)} \right), \ a \in Vert\left(\Sigma^*\right) \text{ with } a \succ 0, \text{ or } -1.$$

For each strictly positive vertex a, we have an exceptional variety E_a. The exceptional varieties E_a which do not intersect $\widetilde{V}^{(l-1)}(K)$ do certainly not induce a pole. For a vertex a for which E_a intersects $\widetilde{V}^{(l-1)}(K)$, we already proved that $\sigma\left(a\right) - \sum_{j=1}^{l-1} d\left(a, \Gamma_j\right) > 0$ (cf. (4.3)). This complete the description of the possible poles of $Z_\Phi(\omega, V^{(l-1)}, f_l)$.

Case II ($b \in T_I(K) \cap \widetilde{V}^{(l-1)}\left(K\right)$ and $b \notin \widetilde{V}^{(l)}\left(K\right)$). In this case $r + l - 1 \leq n$. By using the same reasoning as in the previous case one gets that $I(\omega)$ equals a constant multiplied by (4.4). □

Remark 8. *If in Theorem 2 we assume that $\boldsymbol{f} = (f_1, \ldots, f_l)$, $\boldsymbol{f}\left(0\right) = 0$, is a polynomial mapping, $U = K^n$, and $V^{(l)}\left(K\right)$ and $V^{(l-1)}\left(K\right)$ are convenient and non-degenerate intersection varieties, with $V^{(l-1)}\left(K\right)$ a closed submanifold of K^n. Then the conclusion of Theorem 2 holds without the condition "U is sufficiently small."*

4.1. **The Largest Real Part of the Poles of $Z_\Phi(s, V^{(l-1)}, f_l)$.** Given a $\boldsymbol{f}$: $U \to K^l$, $\boldsymbol{f}\left(0\right) = 0$, an analytic mapping defined on a neighborhood $U \subseteq K^n$ of the origin, and a fixed rational simple polyhedral subdivision $\Sigma^* = \Sigma^*\left(\boldsymbol{f}\right)$ of $\mathbb{R}_+^n$ subordinate to $\Gamma\left(\boldsymbol{f}\right)$, we set

$$Z_\Phi(s, V^{(l-1)}, f_l) := \int\limits_{V^{(l-1)}(K)} \Phi\left(x\right) |f_l(x)|_K^s \mid \gamma_{GL}\left(x\right) \mid ,$$

for $\mathrm{Re}(s) > 0$. As always we will identify $Z_\Phi(s, V^{(l-1)}, f_l)$ with its meromorphic continuation. The correspondence

$$\begin{array}{ccc} S(K^n) & \to & \mathbb{Q}\left(q^{-s}\right) \\ \\ \phi & \to & Z_\Phi(s, V^{(l-1)}, f_l), \end{array}$$

defines a meromorphic distribution on $S(K^n)$. By the poles of $Z_\cdot(s, V^{(l-1)}, f_l)$ we mean the set $\cup_{\phi \in S(K^n)} \left\{ \text{poles of } Z_\phi(s, V^{(l-1)}, f_l) \right\}$. By using the fact that $\boldsymbol{f}(0) = 0$, and that $Z_\phi(s, V^{(l-1)}, f_l)$ can be expressed as a finite sum of Igusa's local zeta functions, it follows from [28, Lemma 2.6] that $Z_\phi(s, V^{(l-1)}, f_l)$ has at least a pole.

We set $\beta_{\boldsymbol{f}}$ to be the largest real part of the poles of $Z_\cdot(s, V^{(l-1)}, f_l)$, and $j_{\boldsymbol{f}}$ to be the maximal order of the poles of $Z_\cdot(s, V^{(l-1)}, f_l)$ having real part $\beta_{\boldsymbol{f}}$. By abuse of language we will say that $\beta_{\boldsymbol{f}}$ is the largest real part of the poles of $Z_\phi(s, V^{(l-1)}, f_l)$. We also set $\gamma_{\boldsymbol{f}}$ to be the maximum of the

$$-\left(\frac{\sigma(a) - \sum_{j=1}^{l-1} d(a, \Gamma_j)}{d(a, \Gamma_l)} \right),$$

where a runs through all the strictly positive vectors in $\mathrm{Vert}(\Sigma^*)$ satisfying $d(a, \Gamma_l) \neq 0$ and $\sigma(a) - \sum_{j=1}^{l-1} d(a, \Gamma_j) > 0$.

Remark 9. *If $\gamma_{\boldsymbol{f}} > -1$, then by Theorem 2, $\gamma_{\boldsymbol{f}} \geq \beta_{\boldsymbol{f}}$ and $j_{\boldsymbol{f}} \leq n - l + 1$.*

If $\boldsymbol{f} = (f_1, \ldots, f_l)$, $\boldsymbol{f}(0) = 0$, is a polynomial mapping, $U = K^n$, and $V^{(l)}(K)$ and $V^{(l-1)}(K)$ are convenient and non-degenerate complete intersection varieties, with $V^{(l-1)}(K)$ a closed submanifold of K^n, then, with the obvious analogous definitions for $\beta_{\boldsymbol{f}}$, $j_{\boldsymbol{f}}$ and $\gamma_{\boldsymbol{f}}$, and deleting the condition "strictly positive" in the definition of $\gamma_{\boldsymbol{f}}$, Remark 9 holds.

The largest real part of the poles of local zeta functions has been studied intensively [4], [5], [8], [24], [28], [29], [30]. In the case $l > 1$ the largest real part of the poles of $Z_\Phi(s, V^{(l-1)}, f_l)$ is not completely determined by the Γ_i.

4.2. **Vanishing of $Z_\Phi(\omega, V^{(l-1)}, f_l)$.** Given $\lambda \in K^\times$, we set

$$V^{(l,\lambda)}(K) := V^{(l,\lambda)} = \left\{ z \in U \mid f_1(z) = \ldots = f_{l-1}(z) = 0,\ f_l(z) = \lambda \right\}.$$

Since any $\omega \in \Omega(K^\times)$ can be expressed as $\omega(z) = \chi(ac\ z)\, |z|_K^s$, $s \in \mathbb{C}$, for $z \in K^\times$, we use the notation $Z_\Phi(\omega, V^{(l-1)}, f_l) := Z_\Phi(s, \chi, V^{(l-1)}, f_l) := Z_\Phi(s, \chi)$.

Theorem 3. *Assume that the $l-$form $\bigwedge_{i=1}^{l} df_i$, with $2 \leq l \leq n$, does not vanish on $V^{(l,\lambda)}$, for any $\lambda \in K^\times$. Then, with the hypotheses of Theorem 2, there exists $e(\Phi) > 0$ in $\mathbb{N}$ such that $Z_\Phi(s, \chi, V^{(l-1)}, f_l) = 0$, for every $s \in \mathbb{C}$, unless the conductor $c(\chi)$ of χ satisfies $c(\chi) \leq e(\Phi)$.*

Proof. By using the proof of Proposition 1, and all the notation introduced there, we have that $Z_\Phi(\omega, V^{(l-1)}, f_l)$ can be expressed as linear combination of classical Igusa's zeta functions, see (2.1). The result follows from Theorem 8.4.1 in [9] by the following assertion.

Claim $\nabla h(y) \neq 0$, for any $y \in$ support of $\Theta \cap \{h(y) \neq 0\}$.

The hypothesis $\bigwedge_{i=1}^{l} df_i \neq 0$ on $V^{(l,\lambda)}$, for any $\lambda \in K^{\times}$, is equivalent to the matrix $\left[\frac{\partial f_i}{\partial x_j}(z)\right]$ has rank l, for any $z \in V^{(l,\lambda)}(K) \setminus V^{(l)}(K)$. We take, as in the proof of Proposition 1, a point $b \in V^{(l-1)}(K)$ and a coordinate system $y = (y_1, \ldots, y_n) = \phi(x)$ around b such that

$$V^{(l-1)}(K) = \{y_i = 0,\ i = 1, \ldots, l-1\},$$

locally, and $h(y) = \left(f_l \circ \phi^{-1}\right)(0, \ldots, 0, y_l, \ldots, y_n)$, then

$$rank_K \left[\frac{\partial f_i}{\partial x_j}\right] = rank_K \begin{bmatrix} I_{(l-1)\times(l-1)} & & O_{(l-1)\times(n-l+1)} \\ & & \\ O_{(1)\times(l-1)} & \ldots & \frac{\partial h}{\partial y_l} \cdots \frac{\partial h}{\partial y_n} \end{bmatrix} = l,$$

where $I_{l-1\times l-1}$ is the identity matrix and $O_{(l-1)\times(n-l+1)}$, $O_{(1)\times(l-1)}$ are zero matrices, at any point of support of $\Theta \cap \{h(y) \neq 0\}$. Hence, at any point $y_0 \in$support of $\Theta \cap \{h(y) \neq 0\}$, there exists $i_0 \in \{l, \ldots, n\}$ such that $\frac{\partial h}{\partial y_{i_0}}(y_0) \neq 0$. $\square$

Remark 10. *The conclusion in Theorem 3 holds, if the condition "the hypotheses of Theorem 2" is replaced by "the hypotheses of Remark 8."*

4.3. **The Oscillatory Integrals $E_{\Phi}(z)$.** In this section we study the asymptotic behavior of the oscillatory integral defined in the introduction:

$$E_{\Phi}(z, V^{(l-1)}, f_l) = E_{\Phi}(z) = \int\limits_{V^{(l-1)}(K)} \Phi(x)\, \Psi(z f_l(x)) \mid \gamma_{GL}(x) \mid,$$

where $z = u\pi^{-m}$, $u \in R_K^{\times}$, $m \in \mathbb{Z}$, and $\Psi(\cdot)$ is the standard additive character on K.

Let $\mathrm{Coeff}_{t^k} Z_{\Phi}(s, \chi)$ denote the coefficient c_k in the power expansion of $Z_{\Phi}(s, \chi)$ in the variable $t = q^{-s}$.

Proposition 2. *Assume that the $l-$form $\bigwedge_{i=1}^{l} df_i$, with $2 \leq l \leq n$, does not vanish on $V^{(l,\lambda)}$, for any $\lambda \in K^{\times}$. Then, with the hypotheses of Theorem 2,*

$$E_{\Phi}\left(u\pi^{-m}\right) = Z_{\Phi}^{(l)}(0, \chi_{triv}) + Coeff_{t^{m-1}} \frac{(t-q)\, Z_{\Phi}(s, \chi_{triv})}{(q-1)(1-t)} +$$
$$\sum_{\chi \neq \chi_{triv}} g_{\chi^{-1}} \chi(u)\, Coeff_{t^{m-c(\chi)}} Z_{\Phi}^{(l)}(s, \chi),$$

where $c(\chi)$ denotes the conductor of χ, and g_{χ} denotes the Gaussian sum

$$g_{\chi} = (q-1)^{-1} q^{1-c(\chi)} \sum_{v \in \left(R_K / P_K^{c(\chi)}\right)^{\times}} \chi(v)\, \Psi\left(v/\pi^{c(\chi)}\right).$$

Proof. The proof uses the same reasoning as the one given by Denef for Proposition 1.4.4 in [2]. $\square$

Remark 11. *The conclusion in Proposition 2 holds, if the condition "the hypotheses of Theorem 2" is replaced by "the hypotheses of Remark 8."*

Theorem 4. *Let $\boldsymbol{f} = (f_1, \ldots, f_l) : U \to K^l$, $\boldsymbol{f}(0) = 0$, $2 \leq l \leq n$, be an analytic mapping defined on a neighborhood $U \subseteq K^n$ of the origin. Assume that $V^{(l)}(K)$ and $V^{(l-1)}(K)$ are germs of convenient and non-degenerate complete intersection varieties, and that $V^{(l-1)}(K)$ is a closed submanifold of U. Fix a rational simple*

polyhedral subdivision $\Sigma^* = \Sigma^*(\boldsymbol{f})$ *of* $\mathbb{R}_+^n$ *subordinate to* $\Gamma(\boldsymbol{f})$. *Assume that* U *is sufficiently small and* Φ *is a Bruhat-Schwartz function whose support is contained in* U. *Assume that the* $l-$*form* $\bigwedge_{i=1}^{l} df_i$ *does not vanish on* $V^{(l,\lambda)}(K)$, *for any* $\lambda \in K^{\times}$. *Then*
(1) for $|z|_K$ *big enough* $E_\Phi(z)$ *is a finite* $\mathbb{C}-$*linear combination of the functions of the form* $\chi(ac\ z)\,|z|_K^{\lambda}\left(\log_q |z|_K\right)^{j_\lambda}$ *with coefficients independent of* z, *and* $\lambda \in \mathbb{C}$ *a pole of* $\left(1-q^{-s-1}\right) Z_\Phi(s,\chi_{triv})$ *or of* $Z_\Phi(s,\chi)$, $\chi \neq \chi_{triv}$, *and with* $j_\lambda \leq$*(multiplicity of pole* λ -1*). Moreover all the poles* λ *appear effectively in this linear combination;*
(2) if $\gamma_{\boldsymbol{f}} > -1$, *then* $|E_\Phi(z)| \leq C(K)\,|z|_K^{\gamma_{\boldsymbol{f}}}\left(\log_q |z|_K\right)^{n-l}$, *for* $|z|_K$ *big enough, where* $C(K)$ *is a positive constant.*

Proof. (1) The result follows from Theorems 2, 3 and Proposition 2 by writing $Z_\Phi(s,\chi)$ in partial fractions. (2) The estimation follows from the first part and Remark 9. □

Remark 12. *The conclusion in Theorem 4 holds, if the condition "the hypotheses of Theorem 2" is replaced by "the hypotheses of Remark 8."*

5. Congruences and Exponential Sums Along Smooth Algebraic Varieties

For any polynomial g over R_K we denote by $\overline{g}$ the polynomial over $\overline{K}$ obtained by reducing each coefficient of g modulo P_K.

Assume that $f_i(x) \in R_K[x_1,\ldots,x_n]$, $f_i(0)=0$, for $i=1,\ldots,l$, with $2 \leq l \leq n$, and that $V^{(l-1)}(K) = \{x \in K^n \mid f_i(x)=0,\ i=1,\ldots,l-1\}$ is a closed submanifold of dimension $n-l+1$. We will say that $V^{(l-1)}(K)$ is *a smooth* K*-algebraic variety of dimension* $n-l+1$, following the convention introduced in Remark 5.

Since R_K^n is compact,

$$V^{(l-1)}(R_K) := V^{(l-1)}(K) \cap R_K^n = \{x \in R_K^n \mid f_i(x)=0,\ i=1,\ldots,l-1\}$$

is a compact submanifold of dimension $n-l+1$.

Let mod P_K^m denote the canonical homomorphism $R_K^n \to (R_K/P_K^m)^n$, for $m,n \in \mathbb{N}\smallsetminus\{0\}$. We will call the image of $A \subseteq R_K^n$ by mod P_K^m, *the reduction* mod P_K^m *of* A, and it will be denoted as A mod P_K^m.

We set

$$V^{(l-1)}\left(\overline{K}\right) := \left\{\overline{z} \in \overline{K}^n \mid \overline{f_i}(\overline{z})=0,\ i=1,\ldots,l-1\right\}.$$

We will say that $V^{(l-1)}(K)$ *has good reduction* mod P_K if $rank_{\overline{K}}\left[\overline{\frac{\partial f_i}{\partial x_j}}(\overline{z})\right] = l-1$, for every $\overline{z} \in V^{(l-1)}\left(\overline{K}\right)$. In this case, we will say that $V^{(l-1)}(K)$ is *a smooth* K*-algebraic variety of dimension* $n-l+1$ *with good reduction* mod P_K.

We also define for $m \in \mathbb{N}\smallsetminus\{0\}$,

$$V^{(l-1)}(R_K/P_K^m) = \{\widetilde{x} \in (R_K/P_K^m)^n \mid ord(f_i(\widetilde{x})) \geq m,\ i=1,\ldots,l-1\}.$$

We note that "$ord(f_i(x)) \geq m$" is independent of the representative chosen to compute $ord(f_i(\widetilde{x}))$.

Definition 3. *(1) Let* $f_i(x) \in R_K[x_1,\ldots,x_n]$, $f_i(0)=0$, *for* $i=1,\ldots,l$, *with* $2 \leq l \leq n$. *The mapping* $\boldsymbol{f}=(f_1,\ldots,f_l): K^n \to K^l$, *is called strongly non-degenerate with respect to* $(\Gamma_1,\ldots,\Gamma_l)$ *(or simply strongly non-degenerate) over* $\overline{K}$,

if for every positive vector $a \in \mathbb{R}^n$, including the origin, and any

$$\overline{z} \in \left\{ \overline{z} \in \left(\overline{K}^{\times} \right)^n \mid \overline{f}_{1,a}(\overline{z}) = \ldots = \overline{f}_{l,a}(\overline{z}) = 0 \right\},$$

it satisfies that $\mathrm{rank}_{\overline{K}} \left[\frac{\partial f_{i,a}}{\partial x_j} (\overline{z}) \right] = l$. *Analogously we call* $\boldsymbol{f}$ *strongly non-degenerate with respect to* $(\Gamma_1, \ldots, \Gamma_l)$ *at the origin (or simply strongly non-degenerate at the origin) over* $\overline{K}$*, if the same condition is satisfied but only for a strictly positive.*
(2) Let $V^{(j)}(K) = V^{(j)} = \{ z \in K^n \mid f_1(z) = \ldots = f_j(z) = 0 \}$*, for* $j = 1, \ldots, l$*, as before. If the mapping* $\boldsymbol{f} = (f_1, \ldots, f_j)$ *is strongly non-degenerate over* $\overline{K}$*, we will say that* $V^{(j)}$ *is a non-degenerate complete intersection variety over* $\overline{K}$*. If* $\boldsymbol{f} = (f_1, \ldots, f_j)$ *is strongly non-degenerate at the origin over* $\overline{K}$*, we will say that* $V^{(j)}$ *is a non-degenerate complete intersection variety at the origin over* $\overline{K}$*.*

We warn the reader that the main role of the word 'strongly' in the previous definition is to emphasize that we are working with a polynomial mapping and that $U = K^n$.

For $a \in \mathbb{R}_+^n$ we set

$$V_a^{(j)}(R_K/P_K^m) := \{ x \in (R_K/P_K^m)^n \mid ord\,(f_{i,a}(x)) \geq m,\ i = 1, \ldots, j \},$$

and

$$V_a^{(j)}(K) := \{ x \in K^n \mid f_{i,a}(x) = 0,\ i = 1, \ldots, j \},$$

for $j = 1, \ldots, l$. Analogously we define $V_a^{(j)}(\overline{K})$.

5.1. **Some Integrals Involving the Dirac Delta Function.** In this section, all the integrals involving the Dirac Delta function are understood as defined in Gel'fand and Shilov's book [6], see also Section 2.1.1.

Lemma 2. *Let* $f_i(x) \in R_K[x_1, \ldots, x_n]$*,* $f_i(0) = 0$*, for* $i = 1, \ldots, l$*, with* $2 \leq l \leq n$*. Assume that* $V^{(l)}(K)$ *and* $V^{(l-1)}(K)$ *are non-degenerate complete intersection varieties over* $\overline{K}$*. Let* $x_0 \in \left(R_K^{\times} \right)^n$ *be a given point, let* $a \in \mathbb{R}_+^n$*, and let* $m \in \mathbb{N} \setminus \{0\}$*. We set*

$$I(s, x_0, m, a) := \int\limits_{x_0 + \left(P_K^m \right)^n} \delta\left(f_{1,a}(x), \ldots, f_{l-1,a}(x) \right) |f_{l,a}(x)|_K^s \, |dx|, \quad \text{for } \operatorname{Re}(s) > 0.$$

Then $I(s, x_0, m, a)$ *equals*

$$\begin{cases} 0 & \text{if} \quad \widetilde{x_0} \notin V_a^{(l-1)}(R_K/P_K^m) \\ q^{-m(n-l+1)-sk} & \text{if} \quad \begin{array}{l} \widetilde{x_0} \in V_a^{(l-1)}(R_K/P_K^m) \text{ and} \\ k := ord\,(f_{l,a}(x_0)) < m \end{array} \\ q^{-m(s+n-l+1)} \left(\frac{1-q^{-1}}{1-q^{-s-1}} \right) & \text{if} \quad \widetilde{x_0} \in V_a^{(l)}(R_K/P_K^m), \end{cases}$$

where $\widetilde{x_0}$ *denotes the image of* x_0 *in* R_K/P_K^m*.*

Proof. Note that $\widetilde{x_0} \notin V_a^{(l-1)}(R_K/P_K^m)$ implies that $I(s, x_0, m, a) = 0$. We consider the case $\widetilde{x_0} \in V_a^{(l-1)}(R_K/P_K^m)$ and $k = ord\,(f_{l,a}(x_0)) < m$. By the Hensel lemma

we may assume $x_0 \in V_a^{(l-1)}(R_K)$.Then $I(s,x_0,m,a)$ can be expressed as

$$I(s,x_0,m,a) = q^{-mn-sk} \int\limits_{R_K^n} \delta\left(f_{1,a}(x_0+\pi^m x), \ldots, f_{l-1,a}(x_0+\pi^m x)\right)|dx|.$$

By reordering the x_i´s, and using the fact that $(f_1,\ldots,f_{l-1})$ is strongly non-degenerate over $\overline{K}$, we assume that $rank_{\overline{K}}\left[\frac{\partial \overline{f_{i,a}}}{\partial x_j}(\overline{z})\right] = l-1$, for any $\overline{z} \in \left(\overline{K}^{\times}\right)^n$. We set $y=(y_1,\ldots,y_n)=\phi(x)$ with

$$y_i = \begin{cases} \frac{f_{i,a}(x_0+\pi^m x)}{\pi^m}, & i=1,\ldots,l-1 \\ x_i, & i=l,\ldots,n. \end{cases}$$

By using the implicit function theorem (see e.g. [9, Lemma 7.4.3]), one gets that $y=\phi(x)$ is measure-preserving bianalytic mapping of R_K^n onto itself. By using $y=\phi(x)$ as change of variables in $I(s,x_0,m,a)$ one gets

$$I(s,x_0,m,a) = q^{-mn-sk} \int\limits_{R_K^n} \delta(\pi^m y_1,\ldots,\pi^m y_{l-1})|dy| = q^{-m(n-l+1)-sk}.$$

Finally we consider $\widetilde{x_0} \in V_a^{(l-1)}(R_K/P_K^m)$ and $k = ord(f_{l,a}(x_0)) \geq m$. This condition is equivalent to $\widetilde{x_0} \in V_a^{(l)}(R_K/P_K^m)$, and by the Hensel lemma we may assume $x_0 \in V_a^{(l)}(R_K)$. By using a reasoning similar to the previously done, one gets that

$$\begin{aligned} I(s,x_0,m,a) &= q^{-mn} \int\limits_{R_K^n} \delta(\pi^m y_1,\ldots,\pi^m y_{l-1})|\pi^m y_l|^s |dy| \\ &= q^{-mn-ms}\left(\int\limits_{R_K^{l-1}} \delta(\pi^m y_1,\ldots,\pi^m y_{l-1})|dy|\right)\left(\int\limits_{R_K} |y_l|^s |dy_l|\right) \\ &= q^{-m(s+n-l+1)}\left(\frac{1-q^{-1}}{1-q^{-s-1}}\right). \end{aligned}$$

□

5.2. **Polynomial Congruences over Submanifolds.** Along this section we will assume that $V^{(l-1)}(K)$ is a smooth K-algebraic variety of dimension $n-l+1$ with good reduction mod P_K. We set $N_m\left(f_l, V^{(l-1)}\right) := N_m$ as

$$\begin{cases} card\left(\left\{\widetilde{x} \in V^{(l-1)}(R_K) \mod P_K^m \mid ord(f_l(\widetilde{x})) \geq m\right\}\right), & m \geq 1 \\ 1, & m=0. \end{cases}$$

Since $V^{(l-1)}(K)$ has good reduction mod P_K, the Hensel lemma implies that

$$V^{(l-1)}(R_K) \mod P_K^m = V^{(l-1)}(R_K/P_K^m),$$

and then

$$\left\{\widetilde{x} \in V^{(l-1)}(R_K) \mod P_K^m \mid ord\left(f_l\left(\widetilde{x}\right)\right) \geq m\right\}$$

$$= \left\{\widetilde{x} \in \left(R_K/P_K^m\right)^n \mid f_1\left(\widetilde{x}\right) \equiv f_2\left(\widetilde{x}\right) \equiv \ldots \equiv f_l\left(\widetilde{x}\right) \equiv 0 \mod P_K^m\right\}.$$

We associate to the sequence $(N_m)_{m\in\mathbb{N}}$ the Poincaré series $P\left(t, f_l, V_{l-1}\right) := P\left(t\right)$ defined as

$$P\left(t\right) = \sum_{m=0}^{\infty} q^{-m(n-l+1)} N_m t^m.$$

We also set

$$Z\left(s, V^{(l-1)}, f_l\right) := \int_{R_K^n} \delta\left(f_1\left(x\right), \ldots, f_{l-1}\left(x\right)\right) \left|f_l(x)\right|_K^s \mid dx \mid .$$

Lemma 3. *If $t = q^{-s}$, then*

$$P\left(t\right) = \frac{1 - tZ\left(s, V^{(l-1)}, f_l\right)}{1-t}. \tag{5.1}$$

Proof. We first note that

$$Z\left(s, V^{(l-1)}, f_l\right) = \int_{R_K^n} \delta\left(f_1\left(x\right), \ldots, f_{l-1}\left(x\right)\right) \left|f_l(x)\right|_K^s \mid dx \mid =$$

$$\sum_{m=0}^{\infty} q^{-ms} \left\{ \int_{\left\{x\in R_K^n \mid ord(f_l(x))\geq m\right\}} \delta\left(f_1\left(x\right), \ldots, f_{l-1}\left(x\right)\right) |dx| \right\}$$

$$- \sum_{m=0}^{\infty} q^{-ms} \left\{ \int_{\left\{x\in R_K^n \mid (ord f_l(x))\geq m+1\right\}} \delta\left(f_1\left(x\right), \ldots, f_{l-1}\left(x\right)\right) |dx| \right\}.$$

The result follows from the previous identity by using the following claim.

Claim 1

$$\int_{\left\{x\in R_K^n \mid ord(f_l(x))\geq m\right\}} \delta\left(f_1\left(x\right), \ldots, f_{l-1}\left(x\right)\right) |dx| = q^{-m(n-l+1)} N_m.$$

The previous integral is equal to a finite sum of integrals of the form

$$I(x_0, m) := \int_{\left\{x\in x_0+\left(P_K^m\right)^n \mid ord(f_l(x))\geq m\right\}} \delta\left(f_1\left(x\right), \ldots, f_{l-1}\left(x\right)\right) |dx|$$

$$= q^{-mn} \int_{\left\{x\in R_K^n \mid ord(f_l(x_0+\pi^m x))\geq m\right\}} \delta\left(f_1\left(x_0 + \pi^m x\right), \ldots, f_{l-1}\left(x_0 + \pi^m x\right)\right) |dx| ,$$

where $x_0 \in R_K^n$ runs through a fixed set of representatives of $V^{(l-1)}(R_K) \bmod P_K^m$. We may assume that $x_0 \in V^{(l-1)}(R_K)$. Indeed, we can choose another set of representatives of $V^{(l-1)}(R_K)$ mod P_K^m which are in $V^{(l-1)}(R_K)$ because $V^{(l-1)}(R_K)$ has good reduction mod P_K.

Note that $I(x_0,m)=0$ if $ord\,(f_l\,(x_0))<m$, and if $ord\,(f_l\,(x_0))\geq m$,

$$I(x_0,m)=q^{-mn}\int\limits_{R_K^n}\delta\left(f_1\left(x_0+\pi^m x\right),\ldots,f_{l-1}\left(x_0+\pi^m x\right)\right)|dx|\,.$$

By using the fact that $V^{(l-1)}(K)$ is a smooth K-algebraic variety of dimension $n-l+1$ with good reduction mod P_K, it follows from the implicit function theorem, possibly after reordering the x_i's, that $y=\phi\,(x)$, with

$$y_i:=\begin{cases}\frac{f_i(x_0+\pi^m x)-f_i(x_0)}{\pi^m}, & i=1,\ldots,l-1\\ \\ x_i, & i=l,\ldots,n,\end{cases}$$

is measure-preserving bianalytic mapping of R_K^n onto itself. Therefore

$$I(x_0,m)=q^{-mn}\int\limits_{R_K^{l-1}}\delta\left(\pi^m y_1+f_1\left(x_0\right),\ldots,\pi^m y_{l-1}+f_{l-1}\left(x_0\right)\right)|dy|\,.$$

We now note that $I(x_0,m)=0$ unless $f_i\,(x_0)\equiv 0 \bmod \pi^m$, $i=1,\ldots,l-1$; in this case, a simple change of variables shows that

$$I(x_0,m)=q^{-m(n-l+1)}\int\limits_{(\pi^m R_K)^{l-1}}\delta\left(z_1,\ldots,z_{l-1}\right)|dz|=q^{-m(n-l+1)}.$$

The Claim follows by observing that there are N_m integrals of type $I(x_0,m)$ each of them equals $q^{-m(n-l+1)}$. □

Theorem 5. *Assume that $V^{(l-1)}(K)$ is a smooth K-algebraic variety of dimension $n-l+1$ with good reduction mod P_K, and that $V^{(l)}(K)$ and $V^{(l-1)}(K)$ are non-degenerate complete intersection varieties over $\overline{K}$. Fix a rational simple polyhedral subdivision $\Sigma^*=\Sigma^*(f_1,\ldots,f_l)$ of $\mathbb{R}_+^n$ subordinate to $\Gamma(f_1,\ldots,f_l)$. Then (1) $P\,(t)$ is a rational function of t with rational coefficients; (2) if $\gamma_{\boldsymbol{f}}>-1$, then*

$$N_m\leq C\,(K)\,q^{m(n-l+1+\gamma_{\boldsymbol{f}})}m^{n-l}.$$

Proof. The first part follows from the rationality of $Z\left(s,V^{(l-1)},f_l\right)$ (cf. Theorem 2 and Remark 8) by (5.1).The second part follows by expanding in simple fractions the left side of (5.1) and using Remark 9. □

5.3. **Exponential Sums Along Smooth Algebraic Varieties.**

Lemma 4. *Let $f_i\,(x)\in R_K\,[x_1,\ldots,x_n]$, $f_i\,(0)=0$, $i=1,\ldots,l$, with $2\leq l\leq n$. Assume that $V^{(l-1)}(K)$ is a smooth algebraic variety of dimension $n-l+1$ with*

good reduction mod P_K. *If* $z = u\pi^{-m}$, $m \in \mathbb{N}$, $u \in R_K^{\times}$, *then*

$$\begin{aligned} E(z) &:= \int_{R_K^n} \delta\left(f_1(x), \ldots, f_{l-1}(x)\right) \Psi\left(z f_l(x)\right) \mid dx \mid \\ &= \int_{V^{(l-1)}(R_K)} \Psi\left(z f_l(x)\right) \mid \gamma_{GL}(x) \mid \\ &= q^{-m(n-l+1)} \sum_{y \in V^{(l-1)}\left(R_K/P_K^m\right)} \Psi\left(z f_l(y)\right). \end{aligned}$$

Proof. The lemma follows from Remark 2 and the following identity:

$$\int_{x_0+\left(P_K^m\right)^n} \delta\left(f_1(x), \ldots, f_{l-1}(x)\right) |dx| =$$

$$\begin{cases} q^{-m(n-l+1)}, & \text{if } x_0 \in V^{(l-1)}\left(R_K\right) \bmod P_K^m \\ \\ 0, & \text{otherwise.} \end{cases}$$

The proof of this identity is very close to the proof of Claim 1 in the proof of Lemma 3. □

Theorem 6. *Let* $f_i(x) \in R_K[x_1, \ldots, x_n]$, $f_i(0) = 0$, *for* $i = 1, \ldots, l$, *with* $2 \leq l \leq n$. *Assume that* $V^{(l-1)}(K)$ *is a smooth algebraic variety of dimension* $n - l + 1$ *with good reduction mod* P_K, *and that* $V^{(l)}(K)$ *and* $V^{(l-1)}(K)$ *are non-degenerate complete intersection varieties. Fix a rational simple polyhedral subdivision* $\Sigma^* = \Sigma^*(f_1, \ldots, f_l)$ *of* $\mathbb{R}_+^n$ *subordinate to* $\Gamma(\boldsymbol{f})$. *Assume that the* $l-$*form* $\bigwedge_{i=1}^{l} df_i$ *does not vanish on* $V^{(l,\lambda)}(K)$, *for any* $\lambda \in K^{\times}$. *If* $\gamma_{\boldsymbol{f}} > -1$, *then*

$$|E(z)| \leq C(K)\, |z|_K^{\gamma_{\boldsymbol{f}}} \left(\log_q |z|_K\right)^{n-l},$$

for $|z|_K$ *big enough, where is* $C(K)$ *a positive constant.*

Proof. The result follows from the previous lemma by applying Theorem 4 (2) and Remark 12. □

6. Explicit Formulas for Local Zeta Functions Supported on Non-degenerate Complete Intersection Varieties

In this section $V^{(l)}(K)$ and $V^{(l-1)}(K)$ are convenient and non-degenerate complete intersection varieties over $\overline{K}$, and $V^{(l-1)}(K)$ is not necessarily a submanifold. We associate to $V^{(l-1)}$and f_l the following local zeta function:

$$\mathcal{Z}\left(s, V^{(l-1)}, f_l\right) := \lim_{r \to +\infty} \int_{R_K^n} \delta_r\left(f_1(x), \ldots, f_{l-1}(x)\right) |f_l(x)|_K^s \mid dx \mid,$$

where $s \in \mathbb{C}$, with $\mathrm{Re}(s) > 0$, and $\mid dx \mid$ is the normalized Haar measure of K^n.

We also define

$$\mathcal{Z}_0\left(s, V^{(l-1)}, f_l\right) := \lim_{r \to +\infty} \int_{(P_K)^n} \delta_r\left(f_1(x), \ldots, f_{l-1}(x)\right) |f_l(x)|_K^s \mid dx \mid,$$

for $\mathrm{Re}(s) > 0$.

The following notation will be used in this section. Given a polynomial mapping $\boldsymbol{f} = (f_1, \ldots, f_l) : K^n \to K^l$ over R_K, and positive vector a, we set as before,

$$V_a^{(j)}\left(\overline{K}\right) = \left\{\overline{z} \in \overline{K}^n \mid \overline{f_{i,a}}\left(\overline{z}\right) = 0,\ i = 1, \ldots, j\right\}.$$

In the case $a = 0$, we take $V_a^{(j)}\left(\overline{K}\right) = V^{(j)}\left(\overline{K}\right)$. Let Δ be a rational simplicial cone spanned by a_i, $i = 1, \ldots, e_\Delta$. We define the *barycenter* of Δ as $b(\Delta) = \sum_{i=1}^{e_\Delta} a_i$.

For $x = (x_1, \ldots, x_n) \in R_K^n$, we define $ord(x) = (ord(x_1), \ldots, ord(x_n)) \in \mathbb{N}^n$, and

$$E_\Delta = \{x \in R_K^n \mid ord(x) \in \Delta\}.$$

Theorem 7. *Let $\boldsymbol{f} = (f_1, \ldots, f_l) : K^n \to K^l$, $\boldsymbol{f}(0) = 0$, $2 \leq l \leq n$, be a convenient polynomial mapping over R_K. Assume that $V^{(l)}(K)$ and $V^{(l-1)}(K)$ are non-degenerate complete intersection varieties over $\overline{K}$.*
(1) There exists a real constant $c(\Gamma_1, \ldots, \Gamma_l)$ such that $\mathcal{Z}\left(s, V^{(l-1)}, f_l\right)$ is holomorphic on $\mathrm{Re}(s) > c(\Gamma_1, \ldots, \Gamma_l)$.
(2) $\mathcal{Z}\left(s, V^{(l-1)}, f_l\right)$ admits a meromorphic continuation to the complex plane as a rational function of q^{-s}, which can be computed from any given rational simplicial polyhedral subdivision Σ^* of $\mathbb{R}_+^n$ subordinate to $\Gamma(\boldsymbol{f})$ as follows:

$$\mathcal{Z}\left(s, V^{(l-1)}, f_l\right) = L_0\left(q^{-s}\right) + \sum_{\Delta \in \Sigma^*} L_\Delta\left(q^{-s}\right) S_\Delta\left(q^{-s}\right),$$

where $L_0(q^{-s})$, $L_\Delta(q^{-s})$, and $S_\Delta(q^{-s})$ are defined as follows:

$$L_0\left(q^{-s}\right) = q^{-(n-l+1)} card\left(V^{(l-1)}\left(\overline{K}\right) \cap \left\{z \in \left(\overline{K}^\times\right)^n \mid \overline{f}_l(z) \neq 0\right\}\right)$$

$$+\ q^{-s-(n-l+1)} card\left(V^{(l)}\left(\overline{K}\right) \cap \left(\overline{K}^\times\right)^n\right)\left(\tfrac{1-q^{-1}}{1-q^{-s-1}}\right);$$

$$L_\Delta\left(q^{-s}\right) =$$

$$q^{-(n-l+1)} card\left(V_{b(\Delta)}^{(l-1)}\left(\overline{K}\right) \cap \left\{z \in \left(\overline{K}^\times\right)^n \mid \overline{f}_{l,b(\Delta)}(z) \neq 0\right\}\right)$$

$$+ q^{-s-(n-l+1)}\left(\tfrac{1-q^{-1}}{1-q^{-s-1}}\right) card\left(V_{b(\Delta)}^{(l)}\left(\overline{K}\right) \cap \left(\overline{K}^\times\right)^n\right);$$

and

$$S_\Delta\left(q^{-s}\right) =$$
$$\left(\textstyle\sum_h q^{d(h,\Gamma_l)s+\sigma(h)-\sum_{j=1}^{l-1} d(h,\Gamma_j)}\right) \prod_{i=1}^{e_\Delta}\left(\frac{q^{-d(a_i,\Gamma_l)s-\sigma(a_i)+\sum_{j=1}^{l-1} d(a_i,\Gamma_j)}}{1-q^{-d(a_i,\Gamma_l)s-\sigma(a_i)+\sum_{j=1}^{l-1} d(a_i,\Gamma_j)}}\right),$$

where h runs through the elements of the set

$$\mathbb{N}^n \cap \left\{\textstyle\sum_{i=1}^{e_\Delta} \mu_i a_i \mid 0 \leq \mu_i < 1 \text{ for } i = 1, \ldots, e_\Delta\right\}.$$

(3) If $V^{(l-1)}(K)$ is a submanifold of K^n, then

$$\mathcal{Z}\left(s, V^{(l-1)}, f_l\right) = Z\left(s, V^{(l-1)}, f_l\right).$$

Proof. We set for $r \in \mathbb{N}$,

$$I_0^{(r)}\left(q^{-s}\right) := \int_{\left(R_K^{\times}\right)^n} \delta_r\left(f_1(x), \ldots, f_{l-1}(x)\right) \left|f_l(x)\right|_K^s |dx|, \ \ \operatorname{Re}(s) > 0,$$

and

$$I_{\Delta}^{(r)}\left(q^{-s}\right) := \int_{E_{\Delta}} \delta_r\left(f_1(x), \ldots, f_{l-1}(x)\right) \left|f_l(x)\right|_K^s |dx|, \ \ \operatorname{Re}(s) > 0.$$

Since $\mathbb{R}_+^n = \{0\} \bigcup \cup_{\Delta \in \Sigma^*} \Delta$, we have

$$\mathcal{Z}\left(s, V^{(l-1)}, f_l\right) = \lim_{r \to +\infty} I_0^{(r)}\left(q^{-s}\right) + \sum_{\Delta \in \Sigma^*} \lim_{r \to +\infty} I_{\Delta}^{(r)}\left(q^{-s}\right).$$

The parts (1)-(2) of the theorem follows from the previous formula by using the following two claims.

Claim 1. $\lim_{r \to +\infty} I_0^{(r)}\left(q^{-s}\right)$ *gives a holomorphic function for* $\operatorname{Re}(s) > -1$. *Furthermore,* $\lim_{r \to +\infty} I_0^{(r)}\left(q^{-s}\right) = L_0\left(q^{-s}\right)$.

Claim 2. *Let* Δ *be a rational simplicial cone spanned by* a_i, $i = 1, \ldots, e_{\Delta}$. *Then* $\lim_{r \to +\infty} I_{\Delta}^{(r)}\left(q^{-s}\right)$ *gives a holomorphic function for* $\operatorname{Re}(s) > c(\Delta)$, *where* $c(\Delta)$ *is a real constant. In addition,*

$$\lim_{r \to +\infty} I_{\Delta}^{(r)}\left(q^{-s}\right) = L_{\Delta}\left(q^{-s}\right) S_{\Delta}\left(q^{-s}\right).$$

Proof Claim 1. Note that $I_0^{(r)}\left(q^{-s}\right)$ can be expressed as a finite sum of integrals of type

$$I^{(r)}\left(s, x_0\right) := \int_{x_0 + \left(P_K\right)^n} \delta\left(f_1(x), \ldots, f_{l-1}(x)\right) \left|f_l(x)\right|_K^s |dx|, \ \ \operatorname{Re}(s) > 0,$$

where x_0 runs through a fixed set of representatives of $\left(R_K^{\times}\right)^n$ mod P_K. We now put

$$y_i := \begin{cases} \frac{f_i\left(x_0 + \pi x\right) - f_i\left(x_0\right)}{\pi} & i = 1, \ldots, l \\ x_i & i = l+1, \ldots, n, \end{cases} \tag{6.1}$$

and $y = \left(y_1, \ldots, y_n\right) := \rho(x)$. Since $\boldsymbol{f} = \left(f_1, \ldots, f_l\right)$ is strongly non-degenerate over $\overline{K}$, $y = \rho(x)$ gives a measure preserving K-bianalytic map from R_K^n to itself (cf. [9, Lemma 7.4.3]), and therefore

$$I^{(r)}\left(s, x_0\right) = q^{-n} \int_{R_K^n} \delta_r\left(\pi y_1 + f_1\left(x_0\right), \ldots, \pi y_{l-1} + f_{l-1}\left(x_0\right)\right) \left|\pi y_l + f_l\left(x_0\right)\right|_K^s |dy|.$$

If $\overline{x_0} \notin V^{(l-1)}\left(\overline{K}\right)$, then $\lim_{r \to +\infty} I^{(r)}\left(s, x_0\right) = 0$. Now, if $\overline{x_0} \in V^{(l-1)}\left(\overline{K}\right)$ and $\overline{x_0} \notin V^{(l)}\left(\overline{K}\right)$, we have

$$\lim_{r \to +\infty} I^{(r)}\left(s, x_0\right) = q^{-n} \lim_{r \to +\infty} \int_{R_K^{l-1}} \delta_r\left(\pi y_1 + f_1\left(x_0\right), \ldots, \pi y_{l-1} + f_{l-1}\left(x_0\right)\right) |dy|$$

$$= q^{-(n-l+1)} \lim_{r\to+\infty} \int_{R_K^{l-1}} \delta_r\left(z_1,\ldots,z_{l-1}\right)|dz| = q^{-(n-l+1)}.$$

If $\overline{x_0} \in V^{(l)}\left(\overline{K}\right)$, we have

$$\lim_{r\to+\infty} I^{(r)}\left(s,x_0\right) = q^{-(n-l+1)-s} \lim_{r\to+\infty} \int_{R_K^{l-1}} \delta_r\left(z_1,\ldots,z_{l-1}\right)|dz| \int_{R_K} |z_l|_K^s\,|dz_l|$$

$$= q^{-(n-l+1)-s} \int_{R_K} |z_l|_K^s\,|dz_l|\,, \text{ for } \operatorname{Re}(s)>0.$$

Therefore,

$$(A)\quad \lim_{r\to+\infty} I^{(r)}\left(s,x_0\right) \text{ gives a holomorphic function for } \operatorname{Re}(s)>0,$$

and

$$(B)\quad \lim_{r\to+\infty} I^{(r)}\left(s,x_0\right) = \begin{cases} 0 & \text{if} \quad \overline{x_0} \notin V^{(l-1)}\left(\overline{K}\right) \\ q^{-(n-l+1)} & \text{if} \quad \begin{cases} \overline{x_0} \in V^{(l-1)}\left(\overline{K}\right) \\ \text{and} \\ \overline{x_0} \notin V^{(l)}\left(\overline{K}\right) \end{cases} \\ q^{-s-(n-l+1)}\left(\frac{1-q^{-1}}{1-q^{-s-1}}\right) & \text{if} \quad \overline{x_0} \in V^{(l)}\left(\overline{K}\right). \end{cases}$$

Now, the announced claim follows from (A) and (B).

Proof of Claim 2. We first note that

$$I_\Delta^{(r)}\left(q^{-s}\right) = \sum_{m\in\mathbb{N}^n\cap\Delta} \int_{ord(x)=m} \delta_r\left(f_1(x),\ldots,f_{l-1}(x)\right)|f_l(x)|_K^s\,|dx|\,.$$

For $m=(m_1,\ldots,m_n)\in\mathbb{N}^n\cap\Delta$ we define

$$x_i = \pi^{m_i}u_i,\ u_i\in R_K^\times,\ i=1,\ldots,n.$$

Then $|dx| = q^{-\sigma(m)}\,|du|$,

$$f_i(x) = \pi^{d(m,\Gamma_i)}\left(f_{i,b(\Delta)}(u)+\pi g_i(u)\right) = \pi^{d(m,\Gamma_i)}\widetilde{f}_i(u)\,,\ i=1,\ldots,l,$$

and $I_\Delta^{(r)}\left(q^{-s}\right)$ equals

$$\sum_{m\in\mathbb{N}^n\cap\Delta} q^{-d(m,\Gamma_l)s-\sigma(m)} \int_{\left(R_K^\times\right)^n} \delta_r\left(\pi^{d(m,\Gamma_1)}\widetilde{f}_1(u),\ldots,\pi^{d(m,\Gamma_{l-1})}\widetilde{f}_{l-1}(u)\right)\left|\widetilde{f}_l(u)\right|_K^s|du|\,.$$

Since

$$\int_{\left(R_K^\times\right)^n} \delta_r\left(\pi^{d(m,\Gamma_1)}\widetilde{f}_1(u),\ldots,\pi^{d(m,\Gamma_{l-1})}\widetilde{f}_{l-1}(u)\right)\left|\widetilde{f}_l(u)\right|_K^s|du|$$

$$= \sum_{x_0\in\left(\mathcal{R}^\times\right)^n} \int_{x_0+\left(P_K\right)^n} \delta_r\left(\pi^{d(m,\Gamma_1)}\widetilde{f}_1(u),\ldots,\pi^{d(m,\Gamma_{l-1})}\widetilde{f}_{l-1}(u)\right)\left|\widetilde{f}_l(u)\right|_K^s|du|\,,$$

where $\mathcal{R}^{\times}$ denotes a fixed set of representatives of $\overline{K}^{\times}$ in R_K, by changing variables as in (6.1), one gets that $\lim_{r\rightarrow+\infty} I_{\Delta}^{(r)}\left(q^{-s}\right)$ is equal to

(6.2)

$$L_{\Delta}\left(q^{-s}\right)\lim_{r\rightarrow+\infty}\sum_{m\in\mathbb{N}^n\cap\Delta}\left(q^{-d(m,\Gamma_l)s-\left(\sigma(m)-\sum_{j=1}^{l-1}d(m,\Gamma_j)\right)}\int_{B_m}\delta_r\left(z_1,\ldots,z_{l-1}\right)|dz|\right),$$

where $B_m=\pi^{d(m,\Gamma_1)+1}R_K\times\ldots\times\pi^{d(m,\Gamma_{l-1})+1}R_K$.

For a strictly positive vertex a, we have $d\left(a,\Gamma_j\right)>0$. A vertex which is not strictly positive is a vector in the set $\{E_1,\ldots,E_n\}$ because $\boldsymbol{f}$ is convenient. But for such a vector E_i, we have $d\left(E_i,\Gamma_j\right)=0$. For strictly positive vertices a_i, we set

$$c(a_i):=-\frac{\sigma\left(a_i\right)-\sum_{j=1}^{l-1}d\left(a_i,\Gamma_j\right)}{d\left(a_i,\Gamma_l\right)},$$

and for a cone Δ spanned by a_i with $i=1,\ldots,e_{\Delta}$, we set

$$c(\Delta):=\max_{i=1,\ldots,e_{\Delta}}\left\{c(a_i)\mid a_i\text{ is strictly positive}\right\}.$$

By using the argument given by Denef and Hoornaert for the case $l=1$ (see [5]), one verifies that series

$$\sum_{m\in\mathbb{N}^n\cap\Delta}q^{-d(m,\Gamma_l)s-\left(\sigma(m)-\sum_{j=1}^{l-1}d(m,\Gamma_j)\right)}$$

converges absolutely on $\mathrm{Re}(s)>c(\Delta)$ and that it defines a holomorphic function on $\mathrm{Re}(s)>c(\Delta)$, and furthermore,

$$S_{\Delta}\left(q^{-s}\right)=\sum_{m\in\mathbb{N}^n\cap\Delta}q^{-d(m,\Gamma_l)s-\left(\sigma(m)-\sum_{j=1}^{l-1}d(m,\Gamma_j)\right)}.$$

By using the dominated convergence theorem, and the fact that series $S_{\Delta}\left(q^{-s}\right)$ converges absolutely on $\mathrm{Re}(s)>c(\Delta)$, one gets that $\lim_{r\rightarrow+\infty} I_{\Delta}^{(r)}\left(q^{-s}\right)$ equals

$$L_{\Delta}\left(q^{-s}\right)\sum_{m\in\mathbb{N}^n\cap\Delta}\left(q^{-d(m,\Gamma_l)s-\left(\sigma(m)-\sum_{j=1}^{l-1}d(m,\Gamma_j)\right)}\lim_{r\rightarrow+\infty}\int_{B_m}\delta_r\left(z_1,\ldots,z_{l-1}\right)|dz|\right)$$
$$=L_{\Delta}\left(q^{-s}\right)S_{\Delta}\left(q^{-s}\right),$$

since $\lim_{r\rightarrow+\infty}\delta_r=\delta$ on $S(K^n)$.

Finally, we set $c\left(\Gamma_1,\ldots,\Gamma_l\right)$ as $\max\left\{\cup_{\Delta\in\Sigma^*}c(\Delta)\cup\{-1\}\right\}$, then $\mathcal{Z}\left(s,V^{(l-1)},f_l\right)$ is holomorphic on $\mathrm{Re}(s)>c\left(\Gamma_1,\ldots,\Gamma_l\right)$.

If $V^{(l-1)}(K)$ is a submanifold of K^n, the previous reasoning shows that the meromorphic continuation of $Z\left(s,V^{(l-1)},f_l\right)$ can be computed from any given rational simplicial polyhedral subdivision of $\mathbb{R}_+^n$ subordinate to $\Gamma\left(\boldsymbol{f}\right)$ using the explicit formula given in the statement of the theorem, then $Z\left(s,V^{(l-1)},f_l\right)=\mathcal{Z}\left(s,V^{(l-1)},f_l\right)$. □

Remark 13. *If in Theorem 7, we assume that $V^{(l)}(K)$ and $V^{(l-1)}(K)$ are non-degenerate complete intersection varieties at the origin over $\overline{K}$, with same notation,*

we have

$$\mathcal{Z}_0\left(s, V^{(l-1)}, f_l\right) = \sum_{\substack{\Delta\in\Sigma^* \\ b(\Delta)\succ 0}} L_\Delta\left(q^{-s}\right) S_\Delta\left(q^{-s}\right).$$

The following problems emerge naturally motivated by our previous theorem.

Problem 2. *Let $V^{(l-1)}(K)$ be K-analytic subset, and let $f_l : V^{(l-1)}(K) \to K$ be an K-analytic function. For which pairs $\left(f_l, V^{(l-1)}\right)$ is $\mathcal{Z}_\Phi\left(\omega, V^{(l-1)}, f_l\right)$ a rational function of q^{-s}?*

Problem 3. *Let $V^{(l-1)}(K)$ be K-analytic submanifold of U, and let $f_l : V^{(l-1)}(K) \to K$ be an K-analytic function. Is $\mathcal{Z}_\Phi\left(\omega, V^{(l-1)}, f_l\right) = Z_\Phi\left(\omega, V^{(l-1)}, f_l\right)$?*

7. Relative Monodromy and Poles of Local Zeta Functions

With the obvious analogous definitions for strongly non-degeneracy over $\mathbb{C}$ (respectively, strongly non-degeneracy at the origin), we have the following. Suppose that $f_1, \ldots, f_l$ are polynomials in n variables with coefficients in a number field F ($\subseteq \mathbb{C}$). Then we can consider $\boldsymbol{f} = (f_1, \ldots, f_l)$ as a map $K^n \to K^l$ for any non-Archimedean completion K of F. If $\boldsymbol{f}$ is strongly non-degenerate over $\mathbb{C}$, then $\boldsymbol{f}$ is strongly non-degenerate over $\overline{K}$ for almost all the completions K of F. This fact follows by applying the Weak Nullstellensatz.

We can associate to a complex non-degenerate polynomial mapping defined over a number field F a local zeta function, say $Z_\Phi(s, \chi, V^{(l-1)}, f_l, K)$. In the case in which $V^{(l-1)}$ is an open subset of K^n, there are several conjectures relating the real parts of the poles of local zeta functions $Z_\Phi(s, \chi, V^{(l-1)}, f_l, K)$ and the eigenvalues of the complex local monodromy (see [2, and references therein]).

Consider two complete intersection varieties defined in a neighborhood U of the origin of $\mathbb{C}^n$:

$$\begin{aligned} V^{(l)}(\mathbb{C}) &= \left\{z \in U \mid f_i(z) = 0,\, i = 1, \ldots, l\right\}, \\ V^{(l-1)}(\mathbb{C}) &= \left\{z \in U \mid f_i(z) = 0,\, i = 1, \ldots, l-1\right\}. \end{aligned}$$

Assume that $V^{(l)}$, $V^{(l-1)}$ are germs of non-degenerate complete intersection varieties at the origin and that $V^{(l-1)}$ has at most an isolated singularity at the origin. Consider the Milnor fibration

$$f_l : E^*(\varepsilon, \delta) \to D_\delta^*,$$

where

$$E^*(\varepsilon, \delta) = \left\{z \in V^{(l-1)} \mid \|z\| < \epsilon,\, 0 < |f_l(z)| \leq \delta\right\},$$

and

$$D_\delta^* = \left\{y \in \mathbb{C} \mid 0 < |y| \leq \delta\right\}.$$

The zeta function of the monodromy of this fibration is called the *l-th principal zeta function of the Milnor fibration* of the mapping $\boldsymbol{f} = (f_1, \ldots, f_l) : (U, 0) \to \left(\mathbb{C}^l, 0\right)$, $2 \leq l \leq n$, and it is denoted as $\zeta_l(t)$ [20]. The corresponding monodromy is called *the l-th principal monodromy of f_l relative to V_{l-1} at the origin.* In [19] M. Oka gave an explicit formula for $\zeta_l(t)$ in terms of the Newton polyhedra $\Gamma_1, \ldots, \Gamma_l$. A similar result was also proved by A. N. Kirillov in [14]. These results are a generalization of Varchenko's explicit formula for the monodromy zeta function of a non-degenerate analytic function [25].

Conjecture 1. *(Relative Monodromy Conjecture) For almost all the completions K of F, if s is a pole of $Z_{\Phi}(s,\chi,f_l,V_{l-1},K)$, then $\exp\left(2\pi\sqrt{-1}\operatorname{Re}(s)\right)$ is an eigenvalue of the l-th principal monodromy of f_l relative to $V^{(l-1)}$ at the origin.*

8. Examples

8.1. **Example.** We set $\boldsymbol{f}=(f_1,f_2)$ with $f_2(x,y,z)=x^8+y^8+z^8+x^2y^2z^2$ and $f_1(x,y,z)=x+y-z$. Then $\boldsymbol{f}$ is strongly non-degenerate over $\mathbb{C}$, and by our previous remarks, $\boldsymbol{f}$ is strongly non-degenerate over $\mathbb{F}_p$, for p big enough. We compute

$$Z_0(s,\boldsymbol{f})=Z_0(s):=\int_{(p\mathbb{Z}_p)^3}|f_2(x,y,z)|_p^s\,\delta(f_1(x,y,z))\,|dxdydz|.$$

Note that $V(K)=\left\{(x,y,z)\in\mathbb{Q}_p^3\mid f_1(x,y,z)=f_2(x,y,z)=0\right\}$ is a hyperplane section, by eliminating z, we get $g(x,y):=x^8+y^8+(x+y)^8+x^2y^2(x+y)^2$ which is a degenerate curve with respect to its Newton polyhedron. Furthermore, by using the fact that $W(K)=\left\{(x,y,z)\in\mathbb{Q}_p^3\mid f_1(x,y,z)=0\right\}$ is a submanifold of $\mathbb{Q}_p^3$, we have

$$Z_0(s)=\int_{(p\mathbb{Z}_p)^2}|g(x,y)|_p^s\,|dxdy|.$$

In the calculation we use the dual diagram of $\Gamma(\boldsymbol{f})$ (i.e., the set of normal vectors to the facets of $\Gamma(\boldsymbol{f})$) and a simple polyhedral subdivision $\Sigma^*(\boldsymbol{f})=\Sigma^*=\{\Delta_i\}_i$ subordinate to $\Gamma(\boldsymbol{f})$ is in figure 1.7.1 of [20, p. 83]

The vertices (i.e., normal vectors to the facets of $\Gamma(\boldsymbol{f})$) are as follows: $E_1=(1,0,0)$, $E_2=(0,1,0)$, $E_3=(0,0,1)$, $P=(1,1,1)$, $P_1=(2,1,1)$, $P_2=(1,2,1)$, $P_3=(1,1,2)$.

For $A\subset\mathbb{R}_+^3$, we set $E_A=\left\{(x,y,z)\in\mathbb{Z}_p^3\mid(ord(x),ord(y),ord(z))\in A\right\}$ as before. Since we have a disjoint union $(\mathbb{R}_{>0})^3=\cup_{\substack{\Delta\in\Sigma^*\\ b(\Delta)\succ 0}}\Delta$, we have

$$\begin{aligned}Z_0(s,\boldsymbol{f})&=\sum_{\substack{\Delta_i\in\Sigma^*\\ b(\Delta_i)\succ 0}}\int_{E_{\Delta_i}}|f_2(x,y,z)|_p^s\,\delta(f_1(x,y,z))\,|dxdydz|\\&=:\sum_{i=1}^{25}Z_{0,\Delta_i}(s,\boldsymbol{f}),\end{aligned}$$

where the cones are defined below.

Claim 1. *If Δ_i has dimension 3, then $Z_{0,\Delta_i}(s,\boldsymbol{f})=0$.*

We consider first the particular case of the cone Δ_1 spanned by E_1, P_1, E_3. In this case E_{Δ_1} equals

$$\bigcup_{a,b,c\in\mathbb{N}\smallsetminus\{0\}}\left\{(x,y,z)\in\mathbb{Z}_p^3\mid x=p^{a+2b}u,y=p^bv,z=p^{b+c}w,\ u,v,w\in\mathbb{Z}_p^\times\right\}.$$

Then $Z_{0,\Delta_1}(s,\boldsymbol{f})$ can be expressed as

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=1}^{\infty}\int_{p^{a+2b}\mathbb{Z}_p^\times\times p^b\mathbb{Z}_p^\times\times p^{b+c}\mathbb{Z}_p^\times}|f_2(x,y,z)|_p^s\,\delta(f_1(x,y,z))\,|dxdydz|.$$

By taking

$$\begin{cases}x=p^{a+2b}u\\ y=p^bv\\ z=p^{b+c}w\end{cases},\quad|dxdydz|=p^{a-4b-c}\quad|dudvdw|\ ,$$

with $u, v, w \in \mathbb{Z}_p^{\times}$ as a change of variables in the previous integral, $Z_{\Delta_1}(s, \boldsymbol{f})$ becomes

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=1}^{\infty} p^{-a-4b-c-8bs} \int\limits_{(\mathbb{Z}_p^{\times})^3} \left|\widetilde{f}_2(u,v,w)\right|_p^s \delta\left(p^b \widetilde{f}_1(u,v,w)\right) |dudvdw|,$$

where $\widetilde{f}_1(u,v,w) = v + p^{a+b}u - p^c w$, $\widetilde{f}_2(u,v,w) = v^8 + p^{8a+8b}u^8 + p^{8c}w^8 + p^{2a+2c}u^2v^2w^2$. Since

$$\delta\left(p^b \widetilde{f}_1(u,v,w)\right) = 0 \text{ for every } (u,v,w) \in \left(\mathbb{Z}_p^{\times}\right)^3,$$

we conclude that $Z_{\Delta_1}(s, \boldsymbol{f}) = 0$.

Let Δ_i be an arbitrary cone of dimension three. For any $a \in \Delta_i$, $F(a, \Gamma_1)$ consists of a monomial, say x, then $\widetilde{f}_1(u,v,w) = u + p^{c(a)}u - p^{c'(a)}w$, with $c(a)$, $c'(a) \in \mathbb{N} \smallsetminus \{0\}$. Note that the only monomials we obtain are x, y, and z. We now use the previous argument to conclude that $Z_{0,\Delta_i}(s, \boldsymbol{f}) = 0$.

Claim 2. *If Δ_i is a cone of dimension two, then the possible values of $Z_{0,\Delta_i}(s, \boldsymbol{f})$ are as follow:*

Cones	Generators	$Z_{0,\Delta_i}(s, \boldsymbol{f})$
Δ_{10}	E_3, P_1	0
Δ_{11}	E_1, P_1	$p^{-1}\left(1-p^{-1}\right)\left(\frac{p^{-3-8s}}{1-p^{-3-8s}}\right)$
Δ_{12}	P_1, P_3	0
Δ_{13}	P_3, E_3	$p^{-1}\left(1-p^{-1}\right)\left(\frac{p^{-3-8s}}{1-p^{-3-8s}}\right)$
Δ_{14}	P_3, E_2	0
Δ_{15}	P_3, P_2	0
Δ_{16}	P_3, P	$\left(1-p^{-1}\right)^2\left(\frac{p^{-3-8s}}{1-p^{-3-8s}}\right)\left(\frac{p^{-2-6s}}{1-p^{-2-6s}}\right)$
Δ_{17}	P, P_1	$\left(1-p^{-1}\right)^2\left(\frac{p^{-3-8s}}{1-p^{-3-8s}}\right)\left(\frac{p^{-2-6s}}{1-p^{-2-6s}}\right)$
Δ_{18}	P_1, P_2	0
Δ_{19}	P, P_2	$\left(1-p^{-1}\right)^2\left(\frac{p^{-3-8s}}{1-p^{-3-8s}}\right)\left(\frac{p^{-2-6s}}{1-p^{-2-6s}}\right)$
Δ_{20}	P_2, E_1	0
Δ_{21}	P_2, E_2	$p^{-1}\left(1-p^{-1}\right)\left(\frac{p^{-3-8s}}{1-p^{-3-8s}}\right)$

By using the same reasoning as in the calculation of $Z_{0,\Delta_1}(s, \boldsymbol{f})$, we have $Z_{0,\Delta_i}(s, \boldsymbol{f}) = 0$, for $i = 10, 12, 14, 18, 20$. We now compute $Z_{0,\Delta_{11}}(s, \boldsymbol{f})$. By using the same reasoning and notation as in the calculation of $Z_{0,\Delta_1}(s, \boldsymbol{f})$, one gets the following expansion for $Z_{0,\Delta_{11}}(s, \boldsymbol{f})$:

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty} p^{-a-4b-8bs} \int\limits_{(\mathbb{Z}_p^{\times})^3} \left|\widetilde{f}_2(u,v,w)\right|_p^s \delta\left(p^b \widetilde{f}_1(u,v,w)\right) |dudvdw|,$$

where $\widetilde{f}_1(u,v,w) = v - w + p^{a+b}u$, $\widetilde{f}_2(u,v,w) = v^8 + w^8 + p^{8a+8b}u^8 + p^{2a}u^2v^2w^2$. Since

$$\delta\left(p^b \widetilde{f}_1(u,v,w)\right) = 0 \text{ for } (u,v,w) \notin \left\{(u,v,w) \in \left(\mathbb{Z}_p^{\times}\right)^3 \mid w = v + p^{a+b}u\right\} := Y,$$

and $\left|\left|\widetilde{f}_2\left(u,v,w\right)\right|_p^s\right|_Y = 1$, and assuming $p>2$, one gets

$$Z_{0,\Delta_{11}}\left(s,\boldsymbol{f}\right)=\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}p^{-a-4b-8bs}\int\limits_{\left(\mathbb{Z}_p^{\times}\right)^3}\delta\left(p^b\widetilde{f}_1\left(u,v,w\right)\right)\left|dudvdw\right|.$$

Fix a set of representatives $\mathcal{R}$ of $\left(\mathbb{F}_p^{\times}\right)^3$ in $\left(\mathbb{Z}_p^{\times}\right)^3$. Then

$$\int\limits_{\left(\mathbb{Z}_p^{\times}\right)^3}\delta\left(p^b\widetilde{f}_1\left(u,v,w\right)\right)\left|dudvdw\right|$$

is a finite sum of integrals of type

$$\begin{aligned}&\textstyle\int_{\xi+p\mathbb{Z}_p^3}\delta\left(p^b\widetilde{f}_1\left(u,v,w\right)\right)\left|dudvdw\right|\\&=p^{-3}\textstyle\int_{\mathbb{Z}_p^3}\delta\left(p^b\widetilde{f}_1\left(\xi_1+pu,\xi_2+pv,\xi_3+pw\right)\right)\left|dudvdw\right|,\end{aligned}$$

where $\xi=\left(\xi_1,\xi_2,\xi_3\right)\in\mathcal{R}$, and $\overline{\xi}\in\left(\mathbb{F}_p^{\times}\right)^3$. By taking

$$\left\{\begin{array}{l}z_1=\frac{\widetilde{f}_1\left(\xi_1+pu,\xi_2+pv,\xi_3+pw\right)-\widetilde{f}_1\left(\xi_1,\xi_2,\xi_3\right)}{p}\\z_2=u\\z_3=w\end{array}\right.,$$

as a change of variables in the previous integral, and using the implicit function theorem (see e.g. [9, Lemma 7.4.3]) one gets

$$p^{-3}\textstyle\int_{\mathbb{Z}_p}\delta\left(p^b\left(pz_1+\widetilde{f}_1\left(\xi_1,\xi_2,\xi_3\right)\right)\right)\left|dz_1\right|.$$

If $\overline{\widetilde{f}_1\left(\xi_1,\xi_2,\xi_3\right)}\neq0$, then the integral is zero. Now if $\overline{\widetilde{f}_1\left(\xi_1,\xi_2,\xi_3\right)}=0$ (note that this equation has $\left(p-1\right)^2$ solutions in $\left(\mathbb{F}_p^{\times}\right)^3$), the integral becomes

$$p^{-3}\textstyle\int_{\mathbb{Z}_p}\delta\left(p^{b+1}z_1\right)\left|dz_1\right|=p^{-2+b}.$$

Therefore

$$\begin{aligned}Z_{0,\Delta_{11}}\left(s,\boldsymbol{f}\right)&=p^{-2}\left(p-1\right)^2\sum_{b=1}^{\infty}p^{-a-3b-8bs}\\&=p^{-1}\left(1-p^{-1}\right)\frac{p^{-3-8s}}{1-p^{-3-8s}}.\end{aligned}$$

The other integrals are calculated in a similar form, as well as the following ones.

Claim 3. *If Δ_i is a cone of dimension one, then the possible values of $Z_{0,\Delta_i}\left(s,\boldsymbol{f}\right)$ are as follow:*

Cones	Generators	$Z_{0,\Delta_i}(s,\boldsymbol{f})$
Δ_{22}	P	$\left(1-p^{-1}\right)^2\left(\frac{p^{-2-2s}}{1-p^{-2-2s}}\right)$
Δ_{23}	P_1	$\left((p-1)^3-N\right)p^{-2}\left(\frac{p^{-3-8s}}{1-p^{-3-8s}}\right)+$ $Np^{-s+2}\left(\frac{1-p^{-1}}{1-p^{-s-1}}\right)\left(\frac{p^{-3-8s}}{1-p^{-3-8s}}\right)$
Δ_{24}	P_2	$\left((p-1)^3-N\right)p^{-2}\left(\frac{p^{-3-8s}}{1-p^{-3-8s}}\right)+$ $Np^{-s+2}\left(\frac{1-p^{-1}}{1-p^{-s-1}}\right)\left(\frac{p^{-3-8s}}{1-p^{-3-8s}}\right)$
Δ_{25}	P_3	$\left((p-1)^3-N\right)p^{-2}\left(\frac{p^{-3-8s}}{1-p^{-3-8s}}\right)+$ $Np^{-s+2}\left(\frac{1-p^{-1}}{1-p^{-s-1}}\right)\left(\frac{p^{-3-8s}}{1-p^{-3-8s}}\right)$

where $N=(p-1)^2\left\{1+\left(\frac{-2}{p}\right)\right\}$, $p>2$, and $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol.

The real parts of the poles of $Z_0(s,\boldsymbol{f})$ belong to the set $\left\{-1,\frac{-3}{8},\frac{-1}{3}\right\}$; $\beta_{\boldsymbol{f}}=\frac{-1}{3}$, and the multiplicity of the corresponding pole is one.

On the other hand, $\varsigma_2(t)=\left(1-t^6\right)\left(1-t^8\right)^3$ (see [20, Example (I-2), pg. 220]). Therefore the relative monodromy conjecture holds in this example.

8.2. **Example.** We set $\boldsymbol{f}=(f_1,f_2)$ with $f_1(x,y)=x^n+y^n$, $f_2(x,y)=x^4+y^4+xy$. Then $\boldsymbol{f}$ is strongly non-degenerate at the origin over $\mathbb{C}$, and by our previous remarks, $\boldsymbol{f}$ is strongly non-degenerate at the origin over $\mathbb{F}_p$, for p big enough. We compute $\mathcal{Z}_0(s,V^{(1)},f_2)$, by using Theorem 7 and Remark 13. We use the simplicial polyhedral subdivision $\Sigma^*(\boldsymbol{f})$ subordinate to $\Gamma(\boldsymbol{f})$ showed in figure 2. The vertices are as follows: $E_1=(0,1)$, $P_1=(1,3)$, $P_2=(1,1)$, $P_3=(3,1)$, $E_2=(1,0)$.

Only the cone generated by P_2 gives a non-zero contribution to the local zeta function,

$$\mathcal{Z}_0\left(s,V^{(1)},f_2\right)=\frac{card\left(\left\{(x,y)\in\left(\mathbb{F}_p^{\times}\right)^2\mid x^n+y^n=0\right\}\right)p^{-2s-3+n}}{1-p^{-2s-2+n}}.$$

Note that the real parts of the poles of $\mathcal{Z}_0\left(s,V^{(1)},f_2\right)$ are zero or positive depending if $n=2$ or if $n>2$.

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